Analytic and Diophantine properties of certain arithmetic Fourier series

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[1] or a number terrorist, as Pieter Moree's colleague says.

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[2]Erich Maria Remarque "Arc de Triomphe"

Manuscripts don't burn.

Mikhail Bulgakov "The Master and Margarita"

Résumé en français

Nous considérons certaines séries de Fourier liées à la théorie des formes modulaires. Nous étudions leurs propriétés analytiques en utilisant deux méthodes différentes. La première revient à trouver et itérer une équation fonctionnelle de la fonction étudiée (méthode d'Itatsu) et la deuxième provient de l'analyse en ondelettes (méthode de Jaffard). Même si les deux méthodes sont différentes, l'étape essentielle de chacune dépend de la modularité sous-jacente. De plus, elles permettent d'obtenir des informations complémentaires.

Pour une matrice $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ et $z \in \mathbb{C}$ on définit la transformation fractionnelle par

$$\gamma \cdot z = \frac{az+b}{cz+d},$$

si $cz+d \in \mathbb{C} \setminus \{0\}$ et $\gamma \cdot \left(-\frac{d}{c}\right) = \infty$. Une fonction holomorphe M_k définie dans le demi-plan supérieur $\mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ est une forme modulaire de poids k sous $SL_2(\mathbb{Z})$ si elle est holomorphe à l'infinité et si pour tout $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ et $z \in \mathbb{H}$, on a

$$M_k(z) = \frac{M_k(\gamma \cdot z)}{(cz+d)^k}. \tag{1.0.1}$$

On écrit $M_k(z) = \sum_{n=0}^{\infty} r_n e^{2\pi i n z}$ pour sa série de Fourier. Une forme modulaire est dite une forme parabolique si $r_0 = 0$. Pour plus de détails, voir par exemple [Ser73]. La première famille de fonctions que nous considérons est définie de la façon suivante. Soit $M_k(z) = \sum_{n=0}^{\infty} r_n e^{2\pi i n z}$, une forme modulaire (ou quasi-modulaire, voir le chapitre 2) sous $SL_2(\mathbb{Z})$ de poids k pair. On définit ensuite

$$M_{k,s}(z) = \sum_{n=1}^{\infty} \frac{r_n}{n^s} \sin(2\pi n z) \quad \text{et} \quad N_{k,s}(z) = \sum_{n=1}^{\infty} \frac{r_n}{n^s} \cos(2\pi n z),$$

pour s convenable tel que $M_{k,s}, N_{k,s}$ soient bien définies et convergent vers une fonction continue sur $\mathbb{R}$. Si M_k est une forme parabolique, on peut prendre $s > \frac{k}{2}$, sinon $s > k$, voir le chapitre 4 lemme 4.8. On suppose que $r_n \in \mathbb{R}$.

La deuxième famille de fonctions étudiée est liée à la fonction thêta. Rappelons que la fonction thêta est définie dans le demi-plan supérieur par $\theta(z) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 z}$ et qu'elle est automorphe de poids $\frac{1}{2}$ sous l'action du groupe $\Gamma_\theta \subset SL_2(\mathbb{Z})$ qui est engendré par $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ et $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ et est d'indice 3. Pour $d \in \mathbb{N}$ et $s > \frac{d}{2}$, nous définissons alors

$$S_{d,s}(x) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \frac{\sin(\pi(n_1^2 + \ldots + n_d^2)x)}{(n_1^2 + \ldots + n_d^2)^s}$$

$$\text{et} \quad T_{d,s}(x) = \sum_{n_1=1}^{\infty} ... \sum_{n_d=1}^{\infty} \frac{\cos(\pi(n_1^2 + ... + n_d^2)x)}{(n_1^2 + ... + n_d^2)^s}.$$

Nous étudions certaines propriétés analytiques de ces séries : la dérivabilité, le module de continuité et l'exposant de Hölder. On dit qu'une fonction réelle f admet un module de continuité g, lorsque pour tout x et y dans le domaine de f on a $|f(x) - f(y)| \leq g(|x-y|)$. On dit qu'une fonction réelle f admet un module de continuité local g en x, lorsque pour tout y dans le domaine de f on a $|f(x) - f(y)| \leq g(|x-y|)$. On dit que $f \in C^\alpha(x_0)$ pour un $\alpha > 0$, $\alpha \notin \mathbb{N}$ s'il existe un polynôme de degré inférieur ou égal à $[\alpha]$, et une constante C (qui peut dépendre de x_0) tels que

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha,$$

lorsque $x \to x_0$. On définit alors l'exposant de Hölder de f en x_0 par $\alpha(x_0) = \sup\{\beta : f \in C^\beta(x_0)\}$. Il est important de noter que si $\alpha(x_0) = \alpha$ pour $\alpha \in \mathbb{N}$, cela n'implique pas que f est nécessairement α-fois dérivable en x_0. Par exemple, l'exposant de Hölder de $x \mapsto x\log(x)$ en $x = 0$ est 1, mais la fonction n'est pas dérivable en 0.

Pour toutes les séries étudiées, nous observons que leurs propriétés analytiques aux points irrationnels sont liées aux propriétés diophantiennes de ces points. On définit les notions suivantes. Soit $x \in \mathbb{R} \setminus \mathbb{Q}$ et $(a_n(x))_n \subseteq \mathbb{N}$ la suite des quotients partiels de x, c'est-à-dire

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{...}}}} = [a_0(x); a_1(x), a_2(x), ...].$$

Soit $(\frac{p_n(x)}{q_n(x)})_n$ la suite des réduites de x, c'est-à-dire $\frac{p_n(x)}{q_n(x)} = [a_0(x); a_1(x), a_2(x), ..., a_n(x)]$.

Les réduites peuvent être obtenues des quotients partiels par la formule de récurrence : $p_n(x) = a_n(x)p_{n-1}(x) + p_{n-2}(x)$, $q_n(x) = a_n(x)q_{n-1}(x) + q_{n-2}(x)$ pour $n \geq 0$ et $p_{-1}(x) = 1$, $p_{-2}(x) = 0, q_{-1}(x) = 0, q_{-2}(x) = 1$.

Définition 0.1. Soit $x \in \mathbb{R} \setminus \mathbb{Q}$. On dit que x est Brjuno-carré (*square-Brjuno*) si

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n(x)^2} < \infty.$$

Nous introduisons également deux conditions techniques :

$$\lim_{n\to\infty} \frac{\log(q_{n+4}(x))}{q_n(x)^2} = 0\ ; \tag{0.0.1}$$

$$\lim_{n\to\infty} \frac{\log(q_{n+3}(x))}{q_n(x)^2} = 0, \text{ et } a_n(x) = 1 \text{ seulement pour un nombre fini de } n. \tag{0.0.2}$$

Nous observons que la condition (0.0.1) est satisfaite pour presque tout x, mais la condition (0.0.2) n'est satisfaite pour presque aucun x. Nous montrons au chapitre 2 que la propriété Brjuno-carré et les conditions (0.0.1) et (0.0.2) sont indépendantes. En 1988, Yoccoz a étudié la fonction définie par

$$B_1(x) = \sum_{n=0}^{\infty} xT(x)T^2(x)...T^{n-1}(x)\log\left(\frac{1}{T^n(x)}\right),$$

où $T(x) = \left\{\frac{1}{x}\right\}$ si $x \neq 0$, $T(0) = 0$ et $T^j(x) = T(T^{j-1}(x))$ pour tout $j \geq 2$, maintenant appelée la fonction de Brjuno, voir [Yoc88, MMY97]. Cette série converge si et seulement si

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n(x)} < \infty.$$

Cette condition est appelée la condition de Brjuno et a été introduite par Brjuno dans l'étude des certains problèmes de systèmes dynamiques, voir [Brj71, Brj72]. Les points de convergence sont les nombres de Brjuno. Nous observons que si x n'est pas Brjuno-carré, alors il est de Liouville, c'est-à-dire pour tout $n \in \mathbb{N}$ il existe $p, q \in \mathbb{Z}, q > 1$ tels que $\left|x - \frac{p}{q}\right| < \frac{1}{q^n}$. On en déduit que la mesure de Lebesgue et la dimension de Hausdorff de l'ensemble des nombres irrationnels qui ne sont pas Brjuno-carré sont égales à 0.

Pour tout $n \in \mathbb{N}$, nous définissons $\kappa_n(x)$ par l'égalité $\left|x - \frac{p_n(x)}{q_n(x)}\right| = \frac{1}{q_n(x)^{\kappa_n(x)}}$. Puis, on définit

$$\mu(x) = \limsup_{n\to\infty} \kappa_n(x),$$
$$\nu(x) = \liminf_{n\to\infty} \kappa_n(x).$$

Pour tout $x \in \mathbb{R} \setminus \mathbb{Q}$, on a $\mu(x) \geq \nu(x) \geq 2$ et pour presque tout x, $\nu(x) = \mu(x) = 2$. Si $\mu(x) < \infty$, alors x est Brjuno-carré et il satisfait (0.0.1), ce qui sera démontré au chapitre 2. La fonction $\mu(x)$ est l'exposant d'irrationalité et il est souvent défini comme la borne inférieure des μ tel que $\left|x - \frac{p}{q}\right| < \frac{1}{q^\mu}$ pour un nombre fini de $p, q \in \mathbb{Z}$. Il résulte d'un théorème classique de Jarník et Besicovitch que la dimension de Hausdorff de l'ensemble $\{x \in \mathbb{R} | \mu(x) = \mu\}$ est $\frac{2}{\mu}$, voir par exemple [Fal03, p. 157]. Par ailleurs Sun et Wu ont récemment démontré que la dimension de Hausdorff de l'ensemble $\{x \in \mathbb{R} | \nu(x) = \mu(x) = \nu\}$ est égale à $\frac{1}{\nu}$ pour tout $\nu > 2$, voir [SW14].

Nous définissons aussi la version "paire" de ces exposants. Soient

$$\mu_{\mathrm{e}}(x) = \limsup_{n\to\infty} \{\kappa_n(x) | p_n(x), q_n(x) \text{ ne sont pas tous deux impairs}\},$$
$$\nu_{\mathrm{e}}(x) = \liminf_{n\to\infty} \{\kappa_n(x) | p_n(x), q_n(x) \text{ ne sont pas tous deux impairs}\}.$$

Au chapitre 2, nous montrons que μ_{e} et ν_{e} sont bien définis. En outre, nous présentons plus de détails sur la théorie des fractions continues dans ce chapitre.

0.1 Motivation

Ce travail est motivé par l'étude de la fonction de Riemann définie par

$$S(x)=\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(\pi n^2 x). \tag{0.1.1}$$

À la fin du 19ème siècle, les mathématiciens soupçonnaient que S était une fonction continue nulle part dérivable. Vers 1910, Hardy et Littlewood ont démontré que S n'est dérivable en aucun irrationnel et en aucun rationnel qui n'est pas de la forme $\frac{\text{impair}}{\text{impair}}$ et que S est $C^{3/4}$ nulle part sauf peut-être aux nombres rationnels $\frac{\text{impair}}{\text{impair}}$, [Har16, HL14]. En 1970 dans [Ger70], Gerver a montré que S est dérivable aux nombres rationnels de la forme $\frac{\text{impair}}{\text{impair}}$. Sa démonstration est élémentaire mais longue. En 1981, dans l'article "Differentiability of Riemann's Function" de 4 pages [Ita81], Itatsu a présenté une démonstration alternative des résultats sur la dérivabilité de S aux nombres rationnels. Il a utilisé la liaison entre $S(x)$ et la fonction thêta. Il a étudié la fonction complexe $\mathcal{S}(x)=\sum_{n=1}^{\infty}\frac{1}{n^2\pi i}e^{in^2\pi x}$, dont la partie réelle est $S(x)$. Puis, en exploitant le lien avec la fonction θ et l'identité modulaire de Jacobi satisfaite par θ, il a obtenu une équation fonctionnelle pour $\mathcal{S}$ dont on déduit que pour tout $\frac{p}{q}\in\mathbb{Q}$, non nul, on a

$$\mathcal{S}\Big(\frac{p}{q}+h\Big)-\mathcal{S}\Big(\frac{p}{q}\Big)=R(p,q)p^{-1/2}e^{\pi i h/(4|h|)}|h|^{1/2}\frac{h}{|h|}-\frac{h}{2}+O(|h|^{3/2}),$$

où $R(p,q)$ est une constante qui dépend de p et q et est qui vaut zero si et seulement si p et q sont tous les deux impairs. Itatsu a lu le comportement de S autour des points rationnels de cette équation. En 1991, en utilisant la méthode d'Itatsu, Duistermaat a montré que S est $C^{1/2}$ aux points rationnels qui n'ont pas la forme $\frac{\text{impair}}{\text{impair}}$ et il a aussi trouvé une borne supérieure de l'exposant de Hölder aux points irrationnels, voir [Dui91].

En 1996, Jaffard et Meyer ont montré que S est $C^{3/2}$ aux nombres rationnels $\frac{\text{impair}}{\text{impair}}$. Finalement, Jaffard dans [Jaf96], en utilisant la théorie des ondelettes, a déterminé que l'exposant de Hölder de S au point irrationnel x est égal à

$$\alpha(x)=\frac{1}{2}+\frac{1}{2\mu_{\mathrm{e}}(x)}.$$

Dans la première partie de la thèse, nous exploitons l'approche proposée par Itatsu. Dans la seconde partie, nous utilisons l'approche de Jaffard.

0.2 Présentation des résultats

Soit $k\geq 4$ pair. La série d'Eisenstein de poids k est définie pour $z\in\mathbb{H}$ par

$$E_k(z)=\frac{1}{2\zeta(k)}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{1}{(m+nz)^k}.$$

On demande $k \geq 4$ pair pour avoir une série absolument convergente. Pour $k = 2$ on considère

$$E_2(z) = \frac{3}{\pi^2} \lim_{\varepsilon \searrow 0} \left(\sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m+nz)^2 |m+nz|^\varepsilon} \right) + \frac{3}{\pi \mathrm{Im}(z)}.$$

Pour $k \geq 2$ la série de Fourier de E_k est

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z},$$

où B_k est le k-ième nombre de Bernoulli et $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. Pour $k \geq 4$ la fonction E_k est modulaire de poids k sous l'action de $SL_2(\mathbb{Z})$. De plus, E_2 est quasi-modulaire de poids 2 sous l'action de $SL_2(\mathbb{Z})$, voir [Zag92]. La fonction E_2 peut être regardée comme une intégrale modulaire (ou d'Eichler) sur $SL_2(\mathbb{Z})$ de poids 2 avec la fonction période rationnelle $-\frac{2\pi i}{z}$, voir par exemple [Kno90].

Pour $k \geq 2$ pair et $s > k$, on considère

$$F_{k,s}(x) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s} \sin(2\pi n x) \quad \text{et} \quad G_{k,s}(x) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s} \cos(2\pi n x).$$

Comme $\sigma_{k-1}(n) \leq n^{k-1}\sigma_0(n)$ et $\sigma_0(n) = o(n^\varepsilon)$ pour tout $\varepsilon > 0$ (voir par exemple [Ten95, p. 83]), ces séries convergent uniformément vers des fonctions continues sur $\mathbb{R}$. Les séries de sinus se comportent différemment des séries de cosinus en ce qui concerne la dérivabilité. On a en effet le

Théorème 0.2. *Les fonctions $F_{2,3}$ et $G_{2,3}$ ne sont dérivables en aucun point de $\mathbb{Q}$. Cependant, $G_{2,3}$ est dérivable à droite et à gauche en chaque rationnel, ce qui n'est pas le cas de $F_{2,3}$.*

En chaque point rationnel $\frac{p}{q}$ la dérivée de $G_{2,3}$ "chute" de $\frac{\pi^4}{3q^2}$, où on prend $q = 1$ si $\frac{p}{q} = 0$, *i.e.* si on écrit $G'_{2,3}\left(\frac{p}{q}-\right), G'_{2,3}\left(\frac{p}{q}+\right)$ pour la dérivée à gauche et à droite de $G_{2,3}$ en $\frac{p}{q}$ respectivement, alors $G'_{2,3}\left(\frac{p}{q}-\right) - G'_{2,3}\left(\frac{p}{q}+\right) = \frac{\pi^4}{3q^2}$. Pour les nombres irrationnels, on a le

Théorème 0.3. *(i) Si $x \in \mathbb{R} \setminus \mathbb{Q}$ est Brjuno-carré et satisfait (0.0.1) ou (0.0.2), alors $F_{2,3}$ est dérivable en x. En revanche, si $x \in \mathbb{R} \setminus \mathbb{Q}$ n'est pas Brjuno-carré, alors $F_{2,3}$ n'est pas dérivable en x.*

(ii) Si $x \in \mathbb{R} \setminus \mathbb{Q}$ satisfait (0.0.1) ou (0.0.2), alors $G_{2,3}$ est dérivable en x.

En particulier, $F_{2,3}$ et $G_{2,3}$ sont presque partout dérivables. Nous croyons que (0.0.1) et (0.0.2) sont des conditions techniques et pourraient être supprimées du théorème si bien que $G_{2,3}$ devrait être partout dérivable. La difficulté est expliquée au chapitre 3.

Nous considérons maintenant le module de continuité de $F_{2,3}$ et $G_{2,3}$.

Théorème 0.4. *Pour tout $x \in (0,1) \setminus \mathbb{Q}$ et tout $y \in (0,1)$, on a*

$$|F_{2,3}(x) - F_{2,3}(y)| \leq C_1|x-y| \log\left(\frac{1}{|x-y|}\right) + C_2|x-y|, \tag{0.2.1}$$

et

$$|G_{2,3}(x) - G_{2,3}(y)| \leq C_3|x-y| \log\left(\frac{1}{|x-y|}\right) + C_4|x-y|, \tag{0.2.2}$$

où les constantes C_1, C_2, C_3, C_4 dépendent seulement de x.

Si x est Brjuno-carré et satisfait (0.0.1) ou (0.0.2), alors $C_1 = 0$. Si x satisfait (0.0.1) ou (0.0.2), alors $C_3 = 0$. Cependant, il existe $C_1, C_3 > 0$ et C_2, C_4 absolues telles que (0.2.1) et (0.2.2) sont satisfaits pour tout $x \in (0,1) \setminus \mathbb{Q}$ et tout $y \in (0,1)$.

Nous croyons que nous pourrions prolonger nos résultats sur la dérivabilité de $F_{k,k+1}$ et $G_{k,k+1}$ à tout k pair. Plus précisément, nous formulons la conjecture suivante.

Conjecture 0.5. *Soit $k \in \mathbb{N}^*$ pair.*

(i) Les fonctions $F_{k,k+1}$ et $G_{k,k+1}$ ne sont dérivables en aucun rationnel. Cependant, $G_{k,k+1}$ est dérivable à droite et à gauche en chaque rationnel.

(ii) La fonction $G_{k,k+1}$ est dérivable en chaque irrationnel.

(iii) La fonction $F_{k,k+1}$ est dérivable en $x \in \mathbb{R} \setminus \mathbb{Q}$ si et seulement si

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n(x)^k} < \infty. \tag{0.2.3}$$

Pour démontrer les théorèmes 0.2-0.4, nous utilisons l'approche proposée par Itatsu. Les démonstrations détaillées sont présentées au chapitre 3, où nous donnons aussi des arguments justifiant la conjecture 0.5.

La fonction de Brjuno satisfait une équation fonctionnelle $B_1(x) = -\log(x) + xB_1\left(\frac{1}{x}\right)$ sur $(0,1)$. Marmi, Moussa and Yoccoz ont étudié une version généralisée de la fonction de Brjuno. Ils ont défini un opérateur linéaire $T_\alpha f(x) = x^\alpha f\left(\frac{1}{x}\right)$ et puis considéré l'équation $(1-T_\alpha)B_f = f$ telle que $B_f(x+1) = B_f(x)$, voir [MMY97, MMY06]. La condition "k-Brjuno" dans (0.2.3) revient à étudier cette équation avec $\alpha = k$ et $f(x) = -\log(x)$.

Nous considérons maintenant l'exposant de Hölder. On a

Théorème 0.6. *Soit $k \geq 4$ pair et M_k une forme modulaire de poids k sous $SL_2(\mathbb{Z})$ qui n'est pas parabolique. Pour $x \in \mathbb{R} \setminus \mathbb{Q}$, soit $\alpha_{M_{k,s}}(x)$ l'exposant de Hölder de $M_{k,s}$ en x. On suppose que*

$$s > k + \frac{k}{\nu(x)} - \frac{k}{\mu(x)}. \tag{0.2.4}$$

Alors

$$\alpha_{M_{k,s}}(x) = s - k + \frac{k}{\mu(x)}.$$

Le résultat est vrai si l'on remplace $M_{k,s}$ par $N_{k,s}$.

On remarque que la condition (0.2.4) est satisfaite pour presque tout x et $s > k$. Nous ne savons pas si nous pouvons juste supposer que $s > k$ pour tout $x \in \mathbb{R} \setminus \mathbb{Q}$.

Remarque 0.7. Dans cette thèse nous considérons des fonctions réelles. Si les coefficients r_n d'une forme modulaire M_k sont complexes, alors la théorème 0.6 est vraie pour les fonctions de type $\sum_{n=0}^{\infty} \frac{\mathrm{Re}(r_n)}{n^s} \sin(2\pi nz)$, $\sum_{n=0}^{\infty} \frac{\mathrm{Im}(r_n)}{n^s} \sin(2\pi nz)$, $\sum_{n=0}^{\infty} \frac{\mathrm{Re}(r_n)}{n^s} \cos(2\pi nz)$, $\sum_{n=0}^{\infty} \frac{\mathrm{Im}(r_n)}{n^s} \cos(2\pi nz)$. De plus, $s - k + \frac{k}{\mu(x)}$ est un minorant de l'exposant de Hölder de la fonction complexe $M_{k,s}$ en $x \in \mathbb{R} \setminus \mathbb{Q}$ dans ce cas.

Si $s > \frac{3k}{2}$, alors la condition (0.2.4) est satisfaite pour tout $x \in \mathbb{R} \setminus \mathbb{Q}$. De plus, l'ensemble de nombres rationnels a dimension de Hausdorff égale à 0. Nous pouvons donc décrire le spectre des singularités de $M_{k,s}$ dans le théorème suivant, où nous utilisons la convention standard que l'ensemble vide a dimension de Hausdorff $-\infty$.

Théorème 0.8. *Soit $k \geq 4$ pair, M_k une forme modulaire de poids k sous $SL_2(\mathbb{Z})$ qui n'est pas parabolique et $s > \frac{3k}{2}$. Soit $\alpha_{M_{k,s}}(x)$ l'exposant de Hölder de $M_{k,s}$ en x. Alors*

$$\dim_{\mathrm{H}}\{x \in \mathbb{R} | \alpha_{M_{k,s}}(x) = \alpha\} = \begin{cases} \frac{2}{k}\alpha - \frac{2}{k}s + 2, & si\ \alpha \in \left[s-k, s-\frac{k}{2}\right], \\ 0 \ ou\ -\infty, & sinon. \end{cases}$$

Comme E_k n'est parabolique pour aucun $k \geq 4$ pair, on peut appliquer le théorème 0.6 quand $M_k = E_k$. Comme $F_{k,s}(x) = -\frac{B_k}{2k} M_{k,s}$, les théorèmes 1.6 et 1.8 sont vrais pour $F_{k,s}$ pour tout $k \geq 4$. Nous pouvons également évaluer l'exposant de Hölder de $F_{2,s}$.

On a le

Théorème 0.9. *Pour $x \in \mathbb{R} \setminus \mathbb{Q}$, soit $\alpha_{M_{2,s}}(x)$ l'exposant de Hölder de $F_{2,s}$ en x. On suppose que*

$$s > 2 + \frac{2}{\nu(x)} - \frac{2}{\mu(x)}. \tag{0.2.5}$$

On a alors

$$\alpha_{M_{2,s}}(x) \geq s - 2 + \frac{2}{\mu(x)}.$$

De plus, si pour une infinité de n,

$$a_n(x) \geq 7, \tag{0.2.6}$$

alors

$$\alpha_{M_{2,s}}(x) = s - 2 + \frac{2}{\mu(x)}.$$

Le résultat est vrai si l'on remplace $F_{2,s}$ par $G_{2,s}$.

La condition (0.2.6) est satisfaite pour presque tout x, car la suite des quotients partiels n'est pas bornée pour presque tout x, voir par exemple [Khi64, p. 60]. Ainsi, pour tout $k \geq 2$ et tout $s > k$, on a $\alpha_{M_{k,s}}(x) = s - \frac{k}{2}$ pour presque tout $x \in \mathbb{R}$. D'un autre côté, si $\mu(x) = \infty$, alors la condition (0.2.6) est satisfaite et nous avons l'optimalité dans ce cas également. Il est probable que la condition (0.2.6) pourrait être supprimée.

Nous considérons maintenant les formes paraboliques.

Théorème 0.10. *Soit $k \geq 4$ pair et M_k une forme parabolique de poids k sous $SL_2(\mathbb{Z})$. Pour $x \in \mathbb{R} \setminus \mathbb{Q}$, soit $\alpha_{M_{k,s}}(x)$ l'exposant de Hölder de $M_{k,s}$ en x. On suppose que*

$$s > \frac{k}{2} + 1 + \frac{2}{\nu(x)} - \frac{2}{\mu(x)}. \tag{0.2.7}$$

(i) On a alors

$$\alpha_{M_{k,s}}(x) \geq s - \frac{k}{2} - 1 + \frac{2}{\mu(x)}.$$

(ii) De plus, s'il existe $N \in \mathbb{N}$ tel que pour une infinité de n,

$$a_n(x) = N \tag{0.2.8}$$

et si $\mu(x) = 2$, alors

$$\alpha_{M_{k,s}}(x) = s - \frac{k}{2}.$$

Le résultat est vrai si l'on remplace $M_{k,s}$ par $N_{k,s}$.

Soit $\pi_i(x,n) = \frac{1}{n}|\{1 \leq j \leq n | a_j = i\}|$. Pour presque tout x on a $\lim_{n\to\infty} \pi_i(x,n) = \frac{1}{\log(2)} \log(1+\frac{1}{i(i+2)})$, voir [IK02, p. 225]. La condition (0.2.8) est donc satisfaite pour presque tout x. De plus, $\nu(x) = \mu(x) = 2$ pour presque tout x, alors pour tout M_k parabolique, pour tout $s > \frac{k}{2} + 1$ on a $\alpha_{M_{k,s}}(x) = s - \frac{k}{2}$ pour presque tout x.

On considère le discriminant modulaire Δ de poids 12 :

$$\Delta(z) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2i\pi n z} = \frac{(2\pi)^{12}}{1728}(E_4(z)^3 - E_6(z)^2),$$

où τ est la fonction de Ramanujan. La fonction Δ est parabolique, donc pour tout $s > 6$ la série

$$\Delta_s(x) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \cos(2\pi n x)$$

converge pour tout $x \in \mathbb{R}$. On peut lui appliquer le théorème 0.10.

Corollaire 0.11. *Pour $x \in \mathbb{R} \setminus \mathbb{Q}$, soit $\alpha_{\Delta_s}(x)$ l'exposant de Hölder de Δ_s en x. On suppose que $s > 8$. Alors pour presque tout x on a*

$$\alpha_{\Delta_s}(x) = s - 6.$$

Zagier dans [Zag10] a considéré Δ_s, il a étudié $\sum_{n=1}^{\infty} \frac{\tau(n)}{n^{11}} \cos(2\pi nx)$ (comme une extension d'une *quantum modular form*) et a noté qu'elle est 4 fois mais pas 6 fois continûment dérivable sur $\mathbb{R}$. Par le corollaire 0.11, pour presque tout x, on a $\alpha_{\Delta_{11}}(x) = 5$.

Pour démontrer les théorèmes 0.6-0.10, nous utilisons l'approche proposée par Jaffard dans [Jaf96]. Les démonstrations détaillées sont présentées au chapitre 4.

L'étude de l'exposant de Hölder de $F_{k,s}$ et $G_{k,s}$ permet de démontrer certains cas de la conjecture 0.5. Par le théorème 0.6, on a que pour $k \geq 4$ et $x \in \mathbb{R} \setminus \mathbb{Q}$, si $\frac{1}{\nu(x)} - \frac{1}{\mu(x)} < \frac{1}{k}$, alors l'exposant de Hölder de $F_{k,k+1}$ et $G_{k,k+1}$ en x sont tous deux égaux à $1 + \frac{k}{\mu(x)}$. Si $\mu(x) < \infty$, nous en déduisons que $F_{k,k+1}$ et $G_{k,k+1}$ sont dérivables en x. La condition $\mu(x) < \infty$ implique (0.2.3) et une direction de la conjecture 0.5 (iii) est vraie dans ce cas. Il est également intéressant de noter que pour presque tout x on a $\mu(x) = \nu(x) = 2$, de sorte que la conjecture est démontrée pour presque tout x pour tout $k \geq 4$.

Nous présentons maintenant nos résultats concernant $S_{d,s}$ et $T_{d,s}$.

Théorème 0.12. *Les fonctions $S_{3,2}$ et $T_{3,2}$ ne sont pas dérivables en 0.*

Théorème 0.13. *Les fonctions $S_{3,2}$ et $T_{3,2}$ ne sont dérivables en aucun nombre rationnel $\frac{p}{q}$ tel que p et q ne sont pas tous deux impairs. Cependant, si $p \in 4\mathbb{Z} + 3$, alors $S_{3,2}$ est dérivable à droite en $\frac{p}{q}$ et si $p \in 4\mathbb{Z} + 1$, alors $S_{3,2}$ est dérivable à gauche en $\frac{p}{q}$.*

Théorème 0.14. *Soit $x \in \mathbb{R} \setminus \mathbb{Q}$ tel que $\mu_e(x) > 4$. Alors $S_{3,2}$ et $T_{3,2}$ ne sont pas dérivables en x.*

Nous démontrons les théorèmes 0.12-0.14 par la méthode d'Itatsu au chapitre 3.

Nous étudions ensuite l'exposant de Hölder de ces séries. Notons $\{y\}$ la partie fractionnaire de y. On a le

Théorème 0.15. *Soit $d \in \mathbb{N}^*$. Pour $x \in \mathbb{R} \setminus \mathbb{Q}$, soit $\alpha_{S_{d,s}}(x)$ l'exposant de Hölder de $S_{d,s}$ en x. On suppose que*

$$s > \frac{d}{2} + \frac{d}{2\nu_e(x)} - \frac{d}{2\mu_e(x)}, \tag{0.2.9}$$

et

$$\{s\} < \frac{d\mu_e(x) - d}{2\mu_e(x)}, \tag{0.2.10}$$

alors

$$\alpha_{S_{d,s}}(x) = s - \frac{d}{2} + \frac{d}{2\mu_e(x)}.$$

Le résultat est vrai si l'on remplace $S_{d,s}$ par $T_{d,s}$.

En fait, une analyse détaillée montre que si (0.2.9) est satisfaite et il existe $\varepsilon \geq 0$ tel que $\{s\} < \frac{2d\mu_{\mathrm{e}}(x)+d\varepsilon-2d}{4\mu_{\mathrm{e}}(x)+2\varepsilon}$ (ce qui est moins fort que (0.2.10)), alors $\alpha_{S_{d,s}}(x) \geq s - \frac{d}{2} + \frac{d}{2\mu_{\mathrm{e}}(x)+\varepsilon}$.

Nous considérons maintenant $S_{3,2}$ et $T_{3,2}$. Nous observons que les conditions (0.2.9) et (0.2.10) sont satisfaites pour tout $x \in \mathbb{R}\setminus\mathbb{Q}$ tel que $\mu_{\mathrm{e}}(x) > 6$. On déduit du théorème 0.15 que l'exposant de Hölder de $S_{3,2}$ (et $T_{3,2}$) en tel x vaut $\alpha_{S_{3,2}}(x) = \frac{1}{2} + \frac{3}{2\mu_{\mathrm{e}}(x)}$. En particulier, nous en concluons que $S_{3,2}$ et $T_{3,2}$ sont dérivables en $x \in \mathbb{R} \setminus \mathbb{Q}$ si $\mu_{\mathrm{e}}(x) < 3$ et ne sont pas dérivables en x si $3 < \mu_{\mathrm{e}}(x) < 6$. Le théorème 0.14 traite le cas $\mu_{\mathrm{e}}(x) > 4$, alors nous obtenons le résultat suivant.

Corollaire 0.16. *Soit* $x \in \mathbb{R} \setminus \mathbb{Q}$.

(i) Si $\mu_e(x) > 3$*, alors les fonctions* $S_{3,2}$ *et* $T_{3,2}$ *ne sont pas dérivables en* x.

(ii) Si $\mu_e(x) < 3$*, alors les fonctions* $S_{3,2}$ *et* $T_{3,2}$ *sont dérivables en* x.

Si $s > \frac{3d}{4}$ alors (0.2.9) est satisfaite pour tout $x \in \mathbb{R}\setminus\mathbb{Q}$ et si $d \geq 4$, la condition (0.2.10) est satisfaite pour tout $x \in \mathbb{R} \setminus \mathbb{Q}$. Nous décrivons le spectre des singularités de $S_{d,s}$ dans ce cas.

Théorème 0.17. *Soit* $d \in \mathbb{N}, d \geq 4$. *Soit* $s > \frac{3d}{4}$*, alors*

$$\dim_{\mathrm{H}}\{x \in \mathbb{R} | \alpha_{S_{d,s}}(x) = \alpha\} = \begin{cases} \frac{4}{d}\alpha - \frac{4}{d}s + 2, & \textit{si } \alpha \in \left[s - \frac{d}{2}, s - \frac{d}{4}\right], \\ 0 \textit{ ou } -\infty, & \textit{sinon.} \end{cases}$$

Les théorèmes 0.15 et 0.17 sont démontrés au chapitre 4.

La fonction de Riemann S correspond à $S_{1,1}$ et des résultats plus précis que les théorèmes 0.15 et 0.17 ont été obtenus par Jaffard dans [Jaf96]. Les détails de cette étude sont présentés au chapitre 4. Dans sa thèse Oppenheim [Opp97] a utilisé la théorie des ondelettes dans son travail sur la régularité d'un analogue bidimensionnel de la série de Riemann (0.1.1). Plus précisément, il a considéré $\mathcal{S}_\alpha(x, y) = \sum_{\substack{m,n\geq 0 \\ (m,n)\neq(0,0)}} \frac{1}{(m^2+n^2)^\alpha} e^{i\pi(m^2x+n^2y)}$ dans le cadre des espaces deux-microlocaux et déterminé la régularité et le spectre des singularités de cette fonction. Chamizo et Ubis dans [CU07] ont étudié une autre généralisation de S : ils ont considéré la fonction $f_{k,s}(x) = \sum_{n=1}^{\infty} \frac{e^{2i\pi n^k x}}{n^s}$, où $\mathcal{S}(x) = f_{2,2}(x)$. Ils ont étudié la dérivabilité, l'exposant de Hölder et le spectre des singularités de $f_{k,s}$ en utilisant des méthodes de théorie des nombres et d'analyse harmonique. Ils ont lié ces concepts à $\mu(x)$ et $\nu(x)$. Puis dans [CU14], Chamizo et Ubis ont étudié le spectre des singularités de fonctions plus générales encore : $f_s(x) = \sum_{n=1}^{\infty} \frac{e^{2i\pi P(n)x}}{n^s}$, où $P \in \mathbb{Z}[x]$. Ils ont introduit de nouvelles méthodes basées sur certaines approximations diophantiennes et des estimations analytiques et arithmétiques fines des sommes exponentielles.

La fonction θ apparaît dans les études des fractions continues. Par exemple, Kraaikamp et Lopes dans [KL96] ont établi la relation entre le groupe Γ_θ et les fractions continues avec les quotients partiels pair. Voir Rivoal et Seuret [RS] pour l'élaboration sur cette connexion pour les fonctions similaires à $S(x)$.

La derivabilité et l'exposant de Hölder des séries de ces deux types ont également été étudiés par Chamizo dans [Cha04]. Dans cet article, il a considéré des séries provenant de formes automorphes $f(x) = \sum_{n=0}^{\infty} r_n e^{2\pi inx}$ de poids k positifs sous l'action d'un groupe fuchsien : $f_s(x) = \sum_{n=1}^{\infty} \frac{r_n}{n^s} e^{2\pi inx}$. Il a démontré que si M_k est une forme parabolique, alors $M_{k,s}$ n'est pas dérivable en x irrationnel si $s < \frac{k}{2} + 1$ et si $\frac{k+1}{2} < s < \frac{k}{2} + 1$, alors $M_{k,s}$ est dérivable en tout nombre rationnel. De plus, on déduit de [Cha04, Théorème 2.1] que l'exposant de Hölder de f_s aux points irrationnels vaut $s - \frac{k}{2}$ pour tout $\frac{k}{2} < s < \frac{k}{2} + 1$. Cependant, sa méthode n'est pas applicable dans le cas de $F_{k,s}$ et $G_{k,s}$ considéré ici, parce qu'il demande que $s < \frac{k}{2} + 1$ pour f non-parabolique, alors que nous considérons $s > k$.

Dans le même article, il a démontré que la fonction $f_{k,s}(x) = \sum_{n=1}^{\infty} \frac{r_k(n)}{n^s} e^{\pi inx}$, définie pour $\frac{k}{2} < s < \frac{k}{2} + 1$, est dérivable en $x = \frac{p}{q}$ si et seulement si p, q sont tous deux impairs, où $r_k(n)$ est le nombre de représentations de n comme une somme de k carrés, où 0 est autorisé, où le signe et l'ordre comptent. Nous observons que $iS_{3,2}(x) + T_{3,2}(x) = \sum_{n=1}^{\infty} \frac{R_3(n)}{n^2} e^{\pi inx}$, où $R_3(n)$ le nombre de représentations de n comme une somme de 3 carrés des nombres strictement positifs et où l'ordre compte. Même si ce n'est pas précisément $f_{3,2}$, le théorème 0.13 est cohérent avec le résultat de Chamizo.

Certains des résultats présentés dans cette thèse ont été rédigés sous forme d'article paru ou de prépublication [Pet13, Pet14, Pet].

La thèse est organisée comme suit. Au chapitre 2 nous rappelons quelques propriétés des fractions continues, des formes modulaires et de la fonction θ, dont nous aurons besoin pour démontrer les théorèmes annoncés. Puis, au chapitre 3 nous discutons de la méthode d'Itatsu, nous démontrons les théorèmes 0.2-0.4, 0.12-0.14 et nous donnons aussi des arguments justifiant la conjecture 0.5. Au chapitre 4 nous discutons de la méthode de Jaffard et nous démontrons les théorèmes 0.6-0.10, 0.15 et 0.17. Enfin, au chapitre 5 nous évoquons des problèmes ouverts qu'il pourrait être intéressant d'étudier dans le futur.

Contents

Chapter 1

Introduction

In this thesis, we consider certain Fourier series which arise from modular or automorphic forms. We study their analytic properties using two different methods. One is based on finding and iterating a functional equation for the function studied (Itatsu's method), the second one comes from wavelet analysis (Jaffard's method). Even though the two methods differ, the crucial steps in both of them are based on the underlined modularity. These methods give complementary information, as we will see later.

For a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and $z \in \mathbb{C}$ we will denote the fractional transformation as

$$\gamma \cdot z = \frac{az+b}{cz+d},$$

if $cz+d \in \mathbb{C} \setminus \{0\}$, and $\gamma \cdot \left(-\frac{d}{c}\right) = \infty$. A holomorphic function M_k defined in the upper-half plane $\mathbb{H} := \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ is called a modular form of weight k for $SL_2(\mathbb{Z})$ if it is holomorphic at infinity and for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$, we have

$$M_k(z) = \frac{M_k(\gamma \cdot z)}{(cz+d)^k}. \tag{1.0.1}$$

We then write $M_k(z) = \sum_{n=0}^{\infty} r_n e^{2\pi i n z}$ for its Fourier series. If $r_0 = 0$, then it is called a cusp form, for details see for example [Ser73]. The first family of functions we consider is defined as follows. Let $M_k(z) = \sum_{n=0}^{\infty} r_n e^{2\pi i n z}$ be a modular (or quasi-modular, see Chapter 2) form of weight k even under $SL_2(\mathbb{Z})$ group. We then define

$$M_{k,s}(z) = \sum_{n=1}^{\infty} \frac{r_n}{n^s} \sin(2\pi n z) \quad \text{and} \quad N_{k,s}(z) = \sum_{n=1}^{\infty} \frac{r_n}{n^s} \cos(2\pi n z),$$

for suitable s such that $M_{k,s}, N_{k,s}$ are well-defined and converge to continuous functions on $\mathbb{R}$. Throughout the thesis we assume that $r_n \in \mathbb{R}$ for all n.

The second family of functions studied arises from the theta function. Recall that the theta function is defined in the upper-half plane by $\theta(z) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 z}$ and it is an automorphic form of weight $\frac{1}{2}$ under the action of the group $\Gamma_\theta \subset SL_2(\mathbb{Z})$ which is the group (of index 3) generated by $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then for $d \in \mathbb{N}$ and $s > \frac{d}{2}$, we define

$$S_{d,s}(x) = \sum_{n_1=1}^{\infty} \dots \sum_{n_d=1}^{\infty} \frac{\sin(\pi(n_1^2 + \dots + n_d^2)x)}{(n_1^2 + \dots + n_d^2)^s}$$

$$\text{and} \quad T_{d,s}(x) = \sum_{n_1=1}^{\infty} ... \sum_{n_d=1}^{\infty} \frac{\cos(\pi(n_1^2 + ... + n_d^2)x)}{(n_1^2 + ... + n_d^2)^s}.$$

We are interested in certain analytic properties of these series, namely differentiability, modulus of continuity and the Hölder regularity exponent. We say that a real-valued function f admits a modulus of continuity g, if for all x, y in the domain of f we have $|f(x) - f(y)| \leq g(|x - y|)$. We say that a real-valued function f admits a local modulus of continuity g at a point x, if for all y in the domain of f we have $|f(x) - f(y)| \leq g(|x - y|)$. We say that $f \in C^{\alpha}(x_0)$ for some $\alpha > 0$, $\alpha \notin \mathbb{N}$ when there exists a polynomial P of degree less than or equal to $[\alpha]$, and a constant C such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^{\alpha},$$

as $x \to x_0$. Then we define the Hölder regularity exponent of f at x_0 as $\alpha(x_0) = \sup\{\beta : f \in C^{\beta}(x_0)\}$. It is important to note that if $\alpha(x_0) = \alpha$ for $\alpha \in \mathbb{N}$, it does not imply that f is α-times differentiable at x_0. For instance, $x \mapsto x\log(x)$ has Hölder exponent 1 at $x = 0$, but it is not differentiable at 0.

In all studied series we find that these analytic properties at irrational points are related to their fine diophantine properties. We make the following definitions. Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $(a_n(x))_n \subseteq \mathbb{N}$ be the sequence of partial quotients of x, that is

$$x = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{...}}}} = [a_0(x); a_1(x), a_2(x), ...].$$

Let $(\frac{p_n(x)}{q_n(x)})_n$ be the sequence of continued fraction approximations of x, that is $\frac{p_n(x)}{q_n(x)} = [a_0(x); a_1(x), a_2(x), ..., a_n(x)]$. The convergents can be obtained from partial quotients by the recurrence relations: $p_n(x) = a_n(x)p_{n-1}(x) + p_{n-2}(x)$, $q_n(x) = a_n(x)q_{n-1}(x) + q_{n-2}(x)$, for $n \geq 0$, and $p_{-1}(x) = 1, p_{-2}(x) = 0, q_{-1}(x) = 0, q_{-2}(x) = 1$.

Definition 1.1. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. We will say that x is a square-Brjuno number if

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n(x)^2} < \infty.$$

In addition, we introduce two technical conditions:

$$\lim_{n\to\infty} \frac{\log(q_{n+4}(x))}{q_n(x)^2} = 0; \tag{$*$}$$

$$\lim_{n\to\infty} \frac{\log(q_{n+3}(x))}{q_n(x)^2} = 0, \text{ and } a_n(x) = 1 \text{ for only finitely many } n. \tag{$**$}$$

We observe that Condition $(*)$ is satisfied for almost all x, but that Condition $(**)$ holds for almost no x. We show in Chapter 2 that the square-Brjuno property and Conditions

$(*)$ and $(**)$ are independent. In 1988, Yoccoz studied the function defined by

$$B_1(x) = \sum_{n=0}^{\infty} xT(x)T^2(x)...T^{n-1}(x) \log\left(\frac{1}{T^n(x)}\right),$$

where T denotes the Gauss map and $T^j(x) = T(T^{j-1}(x))$ for all $j \geq 2$, now called Brjuno function, see [Yoc88, MMY97]. This series converges if and only if

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n(x)} < \infty.$$

This condition is called Brjuno condition and was introduced by Brjuno in the study of certain problems in dynamical systems see [Brj71, Brj72]. The points of convergence are called Brjuno numbers. We note that if an irrational number x is not square-Brjuno, then it must be Liouville, that is for every $n \in \mathbb{N}$ there exist $p, q \in \mathbb{Z}, q > 1$ such that $\left|x - \frac{p}{q}\right| < \frac{1}{q^n}$. It follows that the set of irrational numbers which are not square-Brjuno has both Lebesgue measure and Hausdorff dimension equal to 0.

For each n, we define $\kappa_n(x)$ by the equality $\left|x - \frac{p_n(x)}{q_n(x)}\right| = \frac{1}{q_n(x)^{\kappa_n(x)}}$. We then define

$$\mu(x) = \limsup_{n\to\infty} \kappa_n(x),$$
$$\nu(x) = \liminf_{n\to\infty} \kappa_n(x).$$

For all $x \in \mathbb{R} \setminus \mathbb{Q}$, we have $\mu(x) \geq \nu(x) \geq 2$, and for almost all x, $\nu(x) = \mu(x) = 2$. If $\mu(x) < \infty$, then x is square-Brjuno and it satisfies $(*)$, which we will see in Chapter 2. The function $\mu(x)$ is called the irrationality exponent of x, and it is usually defined as the infimum over μ such that $\left|x - \frac{p}{q}\right| < \frac{1}{q^\mu}$ for only finitely many $p, q \in \mathbb{Z}$. It follows from a classical theorem of Jarník and Besicovitch that the Hausdorff dimension of the set $\{x \in \mathbb{R} | \mu(x) = \mu\}$ is $\frac{2}{\mu}$, see for example [Fal03, p. 157], whereas Sun and Wu recently proved that the Hausdorff dimension of the set $\{x \in \mathbb{R} | \nu(x) = \mu(x) = \nu\}$ is equal to $\frac{1}{\nu}$ for all $\nu > 2$, see [SW14].

We also define an "even" version of these exponents. Let

$$\mu_{\mathrm{e}}(x) = \limsup_{n\to\infty} \{\kappa_n(x) | p_n(x), q_n(x) \text{ are not both odd}\},$$
$$\nu_{\mathrm{e}}(x) = \liminf_{n\to\infty} \{\kappa_n(x) | p_n(x), q_n(x) \text{ are not both odd}\},$$

which are well-defined, as we will see in Chapter 2. We will provide more details on continued fractions in the next chapter.

1.1 Motivation

This work is motivated by the example of the Riemann "non-differentiable" function which is defined as

$$S(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\pi n^2 x). \tag{1.1.1}$$

At the end of the 19th century, S was thought to be continuous but nowhere differentiable. Then in 1910s Hardy and Littlewood proved that $S(x)$ was indeed neither differentiable at any irrational point x, nor at rational points $x = \frac{p}{q}$ such that p, q were not both odd and that was in fact nowhere $C^{3/4}$ except maybe the rational points of the form $\frac{\text{odd}}{\text{odd}}$, [Har16, HL14]. Later, in 1970 in [Ger70], Gerver showed that $S(x)$ was in fact differentiable at rational points $\frac{p}{q}$ such that p and q are both odd, his proof was elementary but long. In 1981, in a 4-page paper "Differentiability of Riemann's Function" [Ita81], Itatsu gave an alternative proof of differentiability of S at these rational points. His method was based on the relationship between $S(x)$ and the theta function θ. He considered a complex-valued function $\mathcal{S}(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 \pi i} e^{in^2 \pi x}$, whose real part is $S(x)$. Then he obtained a functional equation for $\mathcal{S}$ from its relationship to θ and Jacobi identity satisfied by θ, from which he deduced that for all $0 \neq \frac{p}{q} \in \mathbb{Q}$ we have:

$$\mathcal{S}\Big(\frac{p}{q} + h\Big) - \mathcal{S}\Big(\frac{p}{q}\Big) = R(p,q) p^{-1/2} e^{\pi i h/(4|h|)} |h|^{1/2} \frac{h}{|h|} - \frac{h}{2} + O(|h|^{3/2}),$$

where $R(p,q)$ is a constant that depends on p and q and is zero if and only if p and q are both odd. He read off the behaviour of S around rational points from this equation. In 1991, Duistermaat used this method to study Hölder regularity exponent of $S(x)$ reproving the results on its differentiability on $\mathbb{R}$, see [Dui91]. He showed that S is exactly $C^{1/2}$ at the rational points not of the form $\frac{\text{odd}}{\text{odd}}$. Then in 1996, Jaffard and Meyer showed that S is exactly $C^{3/2}$ at the rational points of the form $\frac{\text{odd}}{\text{odd}}$. In the same year Jaffard, using wavelet methods, proved in [Jaf96] that the Hölder regularity exponent of S at an irrational point x is equal to

$$\alpha(x) = \frac{1}{2} + \frac{1}{2\mu_{\mathrm{e}}(x)}.$$

In the first part of the thesis we exploit the approach proposed by Itatsu, in the second part we use the approach of Jaffard in the study of the described families of functions.

1.2 Statement of the results

Let $k \geq 4$ be even. The Eisenstein series of weight k over the upper-half plane $\mathbb{H}$ is defined as

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m+nz)^k}.$$

We require $k \geq 4$ even to have an absolutely convergent sum. For $k = 2$ we consider

$$E_2(z) = \frac{3}{\pi^2} \lim_{\varepsilon \searrow 0} \left(\sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m+nz)^2 |m+nz|^{\varepsilon}} \right) + \frac{3}{\pi \mathrm{Im}(z)}.$$

For $k \geq 2$ the Fourier expansion of E_k is

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z},$$

where B_k is the k-th Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. For all $k \geq 4$, E_k is modular of weight k under the action of $SL_2(\mathbb{Z})$, and E_2 is quasi-modular of weight 2 under the action of $SL_2(\mathbb{Z})$, see for example [Zag92]. The function E_2 can be viewed as a modular (or Eichler) integral on $SL_2(\mathbb{Z})$ of weight 2 with the rational period function $-\frac{2\pi i}{z}$, see for example [Kno90].

For $k \geq 2$ even and $s > k$ we consider

$$F_{k,s}(x) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s} \sin(2\pi n x) \quad \text{and} \quad G_{k,s}(x) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s} \cos(2\pi n x).$$

Since $\sigma_{k-1}(n) \leq n^{k-1}\sigma_0(n)$ and $\sigma_0(n) = o(n^{\varepsilon})$ for all $\varepsilon > 0$ (see for example [Ten95, p. 83]), these series converge on $\mathbb{R}$ to continuous functions. The sine series exhibits different behaviour with respect to differentiability than the cosine series. We have

Theorem 1.2. *Neither $F_{2,3}$ nor $G_{2,3}$ is differentiable at any point in $\mathbb{Q}$. However, $G_{2,3}$ is right and left differentiable at each rational point.*

At each rational $\frac{p}{q}$ the derivative of $G_{2,3}$ drops by $\frac{\pi^4}{3q^2}$, where we take $q = 1$ when $\frac{p}{q} = 0$, *i.e.* if we denote the left and the right derivative of $G_{2,3}$ at $\frac{p}{q}$ by $G'_{2,3}\left(\frac{p}{q}-\right), G'_{2,3}\left(\frac{p}{q}+\right)$ respectively, then $G'_{2,3}\left(\frac{p}{q}-\right) - G'_{2,3}\left(\frac{p}{q}+\right) = \frac{\pi^4}{3q^2}$. For the irrational points, we have

Theorem 1.3. *(i) If $x \in \mathbb{R} \setminus \mathbb{Q}$ is a square-Brjuno number satisfying $(*)$ or $(**)$, then $F_{2,3}$ is differentiable at x. On the other hand, if $x \in \mathbb{R} \setminus \mathbb{Q}$ is not a square-Brjuno number, then $F_{2,3}$ is not differentiable at x.*

(ii) If $x \in \mathbb{R} \setminus \mathbb{Q}$ satisfies $(*)$ or $(**)$, then $G_{2,3}$ is differentiable at x.

In particular, $F_{2,3}$ and $G_{2,3}$ are differentiable almost everywhere. However, we believe that conditions $(*)$ and $(**)$ are technical and could be removed from the theorem and $G_{2,3}$ would be everywhere differentiable. The difficulty is explained in Chapter 3.

Then we consider the modulus of continuity of $F_{2,3}$ and $G_{2,3}$.

Theorem 1.4. *For all $x \in (0,1) \setminus \mathbb{Q}$ and all $y \in (0,1)$, we have*

$$|F_{2,3}(x) - F_{2,3}(y)| \leq C_1|x-y| \log\left(\frac{1}{|x-y|}\right) + C_2|x-y|, \tag{1.2.1}$$

and

$$|G_{2,3}(x) - G_{2,3}(y)| \leq C_3|x-y| \log\left(\frac{1}{|x-y|}\right) + C_4|x-y|, \tag{1.2.2}$$

for some constants C_1, C_2, C_3, C_4 dependent only on x.

If x is square-Brjuno satisfying $(*)$ or $(**)$, then $C_1 = 0$. If x is satisfies $(*)$ or $(**)$, then $C_3 = 0$. However, there exist $C_1, C_3 > 0, C_2, C_4$ absolute such that (1.2.1) and (1.2.2) are satisfied for all $x \in (0,1) \setminus \mathbb{Q}$ and all $y \in (0,1)$.

We believe that we could extend our results on differentiability of $F_{k,k+1}$ and $G_{k,k+1}$ to any even k. Therefore, we formulate the following conjecture.

Conjecture 1.5. *Let $k \in \mathbb{N}^*$ be even. We have the following.*

(i) Neither $F_{k,k+1}$ nor $G_{k,k+1}$ is differentiable at any rational number; however, $G_{k,k+1}$ is right and left differentiable at each rational number.

(ii) The function $G_{k,k+1}$ is differentiable at each irrational number.

(iii) The function $F_{k,k+1}$ is differentiable at $x \in \mathbb{R} \setminus \mathbb{Q}$ if and only if

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n(x)^k} < \infty. \tag{1.2.3}$$

In order to prove Theorems 1.2-1.4, we use the approach proposed by Itatsu. The detailed proofs are presented in Chapter 3, where we also give arguments justifying Conjecture 1.5.

The Brjuno function satisfies a functional equation $B_1(x) = -\log(x) + xB_1\left(\frac{1}{x}\right)$ on $(0,1)$. Marmi, Moussa and Yoccoz studied a generalised version of Brjuno function, namely they define a linear operator $T_\alpha f(x) = x^\alpha f\left(\frac{1}{x}\right)$ and then consider the equation $(1 - T_\alpha)B_f = f$ such that $B_f(x+1) = B_f(x)$, see [MMY97, MMY06]. The "kth-Brjuno condition" in (1.2.3) corresponds to studying this equation with $\alpha = k$ and $f(x) = -\log(x)$.

We now consider Hölder regularity exponents. We have

Theorem 1.6. *Let $k \geq 4$, even, and M_k be a modular form of weight k under $SL_2(\mathbb{Z})$ not a cusp form. For $x \in \mathbb{R} \setminus \mathbb{Q}$, let $\alpha_{M_{k,s}}(x)$ be the Hölder regularity exponent of $M_{k,s}$ at x. Assume that*

$$s > k + \frac{k}{\nu(x)} - \frac{k}{\mu(x)}. \tag{1.2.4}$$

Then,

$$\alpha_{M_{k,s}}(x) = s - k + \frac{k}{\mu(x)}.$$

The same is true if we replace $M_{k,s}$ with $N_{k,s}$.

We note that (1.2.4) is satisfied for almost all x for any $s > k$. We do not know if (1.2.4) can be relaxed to $s > k$ for any $x \in \mathbb{R} \setminus \mathbb{Q}$.

Remark 1.7. In this thesis we are interested in real-valued functions. If we consider a modular form M_k with complex coefficients r_n, then Theorem 1.6 remains valid for the functions of the form $\sum_{n=0}^{\infty} \frac{\mathrm{Re}(r_n)}{n^s} \sin(2\pi nz)$, $\sum_{n=0}^{\infty} \frac{\mathrm{Im}(r_n)}{n^s} \sin(2\pi nz)$, $\sum_{n=0}^{\infty} \frac{\mathrm{Re}(r_n)}{n^s} \cos(2\pi nz)$, $\sum_{n=0}^{\infty} \frac{\mathrm{Im}(r_n)}{n^s} \cos(2\pi nz)$. Moreover, $s - k + \frac{k}{\mu(x)}$ is a lower bound of the Hölder regularity exponent of the complex-valued function $M_{k,s}$ at $x \in \mathbb{R} \setminus \mathbb{Q}$ in this case.

If $s > \frac{3k}{2}$ then Condition (1.2.4) is satisfied for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Also, as the set of rational points has Hausdorff dimension 0, we can describe the spectrum of singularities of $M_{k,s}$ as follows, where we use the standard convention that the empty set has Hausdorff dimension $-\infty$.

Theorem 1.8. *Let $k \geq 4$, even, M_k be a modular form of weight k under $SL_2(\mathbb{Z})$ not a cusp form and $s > \frac{3k}{2}$. Let $\alpha_{M_{k,s}}(x)$ be the Hölder regularity exponent of $M_{k,s}$ at x. Then*

$$\dim_{\mathrm{H}}\{x \in \mathbb{R} | \alpha_{M_{k,s}}(x) = \alpha\} = \begin{cases} \frac{2}{k}\alpha - \frac{2}{k}s + 2, & \text{if } \alpha \in \left[s - k, s - \frac{k}{2}\right], \\ 0 \text{ or } -\infty, & \text{otherwise.} \end{cases}$$

Since E_k is not a cusp form for all $k \geq 4$ even, we can apply Theorem 1.6 if $M_k = E_k$, then $F_{k,s}(x) = -\frac{B_k}{2k} M_{k,s}$, and Theorems 1.6 and 1.8 are valid for $F_{k,s}$ for all $k \geq 4$. We can also evaluate the Hölder regularity exponent of $F_{2,s}$.

We have

Theorem 1.9. *For $x \in \mathbb{R} \setminus \mathbb{Q}$, let $\alpha_{M_{2,s}}(x)$ be the Hölder regularity exponent of $F_{2,s}$ at x. Assume that*

$$s > 2 + \frac{2}{\nu(x)} - \frac{2}{\mu(x)}. \tag{1.2.5}$$

We have

$$\alpha_{M_{2,s}}(x) \geq s - 2 + \frac{2}{\mu(x)}.$$

Furthermore, if for infinitely many n,

$$a_n(x) \geq 7, \tag{1.2.6}$$

then

$$\alpha_{M_{2,s}}(x) = s - 2 + \frac{2}{\mu(x)}.$$

The same is true if we replace $F_{2,s}$ with $G_{2,s}$.

Condition (1.2.6) is satisfied for almost all x, as the sequence of partial quotients is unbounded for almost all x, see for example [Khi64, p. 60]. Therefore, for all $k \geq 2$ and all $s > k$, we have $\alpha_{M_{k,s}}(x) = s - \frac{k}{2}$ for almost all $x \in \mathbb{R}$. On the other hand, if $\mu(x) = \infty$, then Condition (1.2.6) is satisfied, and we obtain the optimality in this case as well. It is likely that Condition (1.2.6) could be removed.

We now consider cusp forms.

Theorem 1.10. *Let $k \geq 4$, even, and M_k be a cusp form of weight k under $SL_2(\mathbb{Z})$. For $x \in \mathbb{R} \setminus \mathbb{Q}$, let $\alpha_{M_{k,s}}(x)$ be the Hölder regularity exponent of $M_{k,s}$ at x. Assume that*

$$s > \frac{k}{2} + 1 + \frac{2}{\nu(x)} - \frac{2}{\mu(x)}. \tag{1.2.7}$$

(i) We have

$$\alpha_{M_{k,s}}(x) \geq s - \frac{k}{2} - 1 + \frac{2}{\mu(x)}.$$

(ii) Moreover, if there exists $N \in \mathbb{N}$ such that for infinitely many n

$$a_n(x) = N, \tag{1.2.8}$$

and if $\mu(x) = 2$, then

$$\alpha_{M_{k,s}}(x) = s - \frac{k}{2}.$$

We get the same results if we replace $M_{k,s}$ with $N_{k,s}$.

Let $\pi_i(x, n) = \frac{1}{n}|\{1 \leq j \leq n | a_j = i\}|$ denote the frequency of appearance of i among the first n partial quotients of x. It is well-known that for almost all x we have $\lim_{n\to\infty} \pi_i(x, n) = \frac{1}{\log(2)} \log(1 + \frac{1}{i(i+2)})$, see [IK02, p. 225]. It shows that Condition (1.2.8) is satisfied for almost all x. Furthermore, $\nu(x) = \mu(x) = 2$ for almost all x, then for all M_k cusp form, for all $s > \frac{k}{2} + 1$ we have $\alpha_{M_{k,s}}(x) = s - \frac{k}{2}$ for almost all x.

Consider the discriminant modular form Δ of weight 12, which can be written

$$\Delta(z) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2i\pi nz} = \frac{(2\pi)^{12}}{1728}(E_4(z)^3 - E_6(z)^2),$$

where τ is the Ramanujan function. Since Δ is a cusp form, for any $s > 6$ the series

$$\Delta_s(x) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \cos(2\pi nx)$$

converges for all $x \in \mathbb{R}$. We apply Theorem 1.10 to it.

Corollary 1.11. *For $x \in \mathbb{R} \setminus \mathbb{Q}$, let $\alpha_{\Delta_s}(x)$ be the Hölder regularity exponent of Δ_s at x. Assume that $s > 8$. Then for almost all x we have*

$$\alpha_{\Delta_s}(x) = s - 6.$$

Zagier in [Zag10] considered series of the type of Δ_s, in particular he studied Δ_{11} (which he regards as an extension of a quantum modular form) and mentioned that it is 4 times but not 6 times continuously differentiable on $\mathbb{R}$. By Corollary 1.11, for almost all x, we have $\alpha_{\Delta_{11}}(x) = 5$.

In order to prove Theorems 1.6-1.10, we use the approach proposed by Jaffard in [Jaf96]. The detailed proofs are presented in Chapter 4.

Studying the Hölder regularity exponents of $F_{k,s}$ and $G_{k,s}$ enables us to prove some cases of Conjecture 1.5. By Theorem 1.6, we have that for $k \geq 4$ and $x \in \mathbb{R} \setminus \mathbb{Q}$, if $\frac{1}{\nu(x)} - \frac{1}{\mu(x)} < \frac{1}{k}$, then the Hölder regularity exponents of $F_{k,k+1}$ and $G_{k,k+1}$ at x are both $1 + \frac{k}{\mu(x)}$. If $\mu(x) < \infty$, then we conclude that both $F_{k,k+1}$ and $G_{k,k+1}$ are differentiable at x. The condition $\mu(x) < \infty$ implies (1.2.3), and we see that one direction of Conjecture 1.5 (iii) is true. It is also worth noting that since for almost all x we have $\mu(x) = \nu(x) = 2$, the conjecture is proved for almost all x for all $k \geq 4$.

We now present results concerning $S_{d,s}$ and $T_{d,s}$.

Theorem 1.12. *Neither $S_{3,2}$ nor $T_{3,2}$ is differentiable at 0.*

Theorem 1.13. *The functions $S_{3,2}$ and $T_{3,2}$ are not differentiable at any rational point $\frac{p}{q}$ such that p and q are not both odd. However, if $p \in 4\mathbb{Z}+3$, then $S_{3,2}$ is right differentiable, and if $p \in 4\mathbb{Z}+1$, then $S_{3,2}$ is left differentiable at $\frac{p}{q}$.*

Theorem 1.14. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $\mu_e(x) > 4$. Then neither $S_{3,2}$ nor $T_{3,2}$ is differentiable at x.*

Again, we prove Theorems 1.12-1.14 by following Itatsu's method in Chapter 3.

We also examine Hölder regularity of these series. Let $\{y\}$ denote the fractional part of y, we then have

Theorem 1.15. *Let $d \in \mathbb{N}^*$. For $x \in \mathbb{R} \setminus \mathbb{Q}$, let $\alpha_{S_{d,s}}(x)$ be the Hölder regularity exponent of $S_{d,s}$ at x. Assume that*

$$s > \frac{d}{2} + \frac{d}{2\nu_e(x)} - \frac{d}{2\mu_e(x)}, \tag{1.2.9}$$

and

$$\{s\} < \frac{d\mu_e(x) - d}{2\mu_e(x)}, \tag{1.2.10}$$

then

$$\alpha_{S_{d,s}}(x) = s - \frac{d}{2} + \frac{d}{2\mu_e(x)}.$$

The same is true if we replace $S_{d,s}$ with $T_{d,s}$.

In fact, a detailed analysis shows that if (1.2.9) is satisfied and there exists $\varepsilon \geq 0$ such that $\{s\} < \frac{2d\mu_e(x)+d\varepsilon-2d}{4\mu_e(x)+2\varepsilon}$ (which is a weaker assumption than (1.2.10)), then $\alpha_{S_{d,s}}(x) \geq s - \frac{d}{2} + \frac{d}{2\mu_e(x)+\varepsilon}$.

Consider now $S_{3,2}$ and $T_{3,2}$. We observe that Conditions (1.2.9) and (1.2.10) are satisfied for all $x \in \mathbb{R} \setminus \mathbb{Q}$ with $\mu_e(x) < 6$. We deduce that the Hölder regularity exponent of $S_{3,2}$ (and $T_{3,2}$) at such an x is equal to $\alpha_{S_{3,2}}(x) = \frac{1}{2} + \frac{3}{2\mu_e(x)}$. In particular, it follows from Theorem 1.15 that $S_{3,2}$ and $T_{3,2}$ are differentiable at $x \in \mathbb{R} \setminus \mathbb{Q}$ if $\mu_e(x) < 3$ and are not differentiable at x if $3 < \mu_e(x) < 6$. Since Theorem 1.14 addresses the case when $\mu_e(x) > 4$, we formulate the following result.

Corollary 1.16. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$.*

(i) If $\mu_e(x) > 3$, then neither $S_{3,2}$ nor $T_{3,2}$ is differentiable at x.

(ii) If $\mu_e(x) < 3$, then $S_{3,2}$ and $T_{3,2}$ are both differentiable at x.

If $s > \frac{3d}{4}$, then (1.2.9) is satisfied for all $x \in \mathbb{R} \setminus \mathbb{Q}$, and when $d \geq 4$, Condition (1.2.10) is satisfied for all $x \in \mathbb{R} \setminus \mathbb{Q}$. We describe the spectrum of singularities of $S_{d,s}$ in this case.

Theorem 1.17. *Let $d \in \mathbb{N}, d \geq 4$. Let $s > \frac{3d}{4}$, then*

$$\dim_{\mathrm{H}}\{x \in \mathbb{R} | \alpha_{S_{d,s}}(x) = \alpha\} = \begin{cases} \frac{4}{d}\alpha - \frac{4}{d}s + 2, & \textit{if } \alpha \in \left[s - \frac{d}{2}, s - \frac{d}{4}\right], \\ 0 \textit{ or } -\infty, & \textit{otherwise.} \end{cases}$$

Theorems 1.15 and 1.17 are proved in Chapter 4.

The Riemann function S corresponds to $S_{1,1}$, and more precise results than in Theorems 1.15 and 1.17 were obtained by Jaffard in [Jaf96]. The details of this work are presented in Chapter 4. Oppenheim in his thesis [Opp97] applied wavelet theory in his study of regularity of a two-dimensional analogue of the Riemann series (1.1.1). Namely, he considered $\mathcal{S}_\alpha(x, y) = \sum_{\substack{m,n\geq 0 \\ (m,n)\neq(0,0)}} \frac{1}{(m^2+n^2)^\alpha} e^{i\pi(m^2x+n^2y)}$ in the context of two-microlocal spaces and determined the regularity and the spectrum of singularities of this function. Chamizo and Ubis in [CU07] considered a different generalisation of S: they considered the function $f_{k,s}(x) = \sum_{n=1}^{\infty} \frac{e^{2i\pi n^k x}}{n^s}$, where $\mathcal{S}(x) = f_{2,2}(x)$. They studied the differentiability, Hölder exponent and spectrum of singularities of $f_{k,s}$ by using methods from number theory and harmonic analysis. They related these concepts to $\mu(x)$ and $\nu(x)$. Then in [CU14] Chamizo and Ubis studied the spectrum of singularities of even more generalised functions, namely $f_s(x) = \sum_{n=1}^{\infty} \frac{e^{2i\pi P(n)x}}{n^s}$, where $P \in \mathbb{Z}[x]$. They introduced new methods based on some special diophantine approximations and fine analytic and arithmetic estimations of exponential sums.

The function θ appears in the study of continued fractions. For example Kraaikamp and Lopes in [KL96] establish the relation between the group Γ_θ and continued fraction with even partial quotients. See Rivoal and Seuret [RS] for an elaboration of this connection for functions similar to $S(x)$.

Differentiability and Hölder regularity of series of these two types were also studied by Chamizo in [Cha04]. In this paper, he studied series arising from automorphic forms $f(x) = \sum_{n=0}^{\infty} r_n e^{2\pi inx}$ of positive weights k under the Fuchsian group with a multiplier system: $f_s(x) = \sum_{n=1}^{\infty} \frac{r_n}{n^s} e^{2\pi inx}$. His method is based on the theory of automorphic forms. He proved that if M_k is a cusp form, then $M_{k,s}$ is not differentiable at any irrational x if $s < \frac{k}{2} + 1$, and if $\frac{k+1}{2} < s < \frac{k}{2} + 1$, then $M_{k,s}$ is differentiable at all rational points. Moreover, it follows from [Cha04, Theorem 2.1] that the Hölder regularity exponent of f_s at irrational points is equal to $s - \frac{k}{2}$ for all $\frac{k}{2} < s < \frac{k}{2} + 1$. However, his method is not applicable in the case of $F_{k,s}$ and $G_{k,s}$ considered here, because he requires $s < \frac{k}{2} + 1$ for f not a cusp form and we consider $s > k$. In the same paper, he also proves that the function $f_{k,s}(x) = \sum_{n=1}^{\infty} \frac{r_k(n)}{n^s} e^{\pi inx}$, where $r_k(n)$ is the number of representations of n as a sum of k squares, where 0 are allowed, the sign and order matter, defined for $\frac{k}{2} < s < \frac{k}{2} + 1$, is differentiable at $x = \frac{p}{q}$ if and only if p, q are both odd. We note that $iS_{3,2}(x) + T_{3,2}(x) = \sum_{n=1}^{\infty} \frac{R_3(n)}{n^2} e^{\pi inx}$, where $R_3(n)$ is the number of representations of n as a sum of 3 squares of strictly positive numbers and the order matters. Even though it is not precisely $f_{3,2}$, Theorem 1.13 is coherent with Chamizo's results.

Some of the results presented in this thesis were published in [Pet13, Pet14, Pet].

Chapter 2
Preliminaries

2.1 Continued fractions

Before we start proving the announced theorems, we will collect and prove some properties of continued fractions. For the introduction to continued fractions see classical textbooks by Hardy and Wright [HW60] and Khinchin [Khi64].

If not otherwise stated, we will write $a_n = a_n(x)$, $p_n = p_n(x)$, $q_n = q_n(x)$, $\kappa_n = \kappa_n(x)$.

Let T be the Gauss map, *ie.* $T(0) = 0$ and $T(x) = \frac{1}{x} \mod 1$ otherwise. For brevity, write $T^0(x) = x$ and $T^k(x) = T(T^{k-1}(x))$ if $k > 0$. The partial quotients of x can be calculated from the Gauss map by $a_i = \left[\frac{1}{T^{i-1}(x)}\right]$, where $[y]$ is the floor function. Firstly, we note that from $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ we get that $q_n p_{n-1} - p_n q_{n-1} = -(q_{n-1}p_{n-2} - p_{n-1}q_{n-2})$. Since $q_0 p_{-1} - p_0 q_{-1} = 1$, we obtain by induction that for all $n \geq 0$

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n. \tag{2.1.1}$$

It follows from (2.1.1) that $\mu_{\mathrm{e}}(x)$ and $\nu_{\mathrm{e}}(x)$ are well-defined.

The convergents satisfy the following.

Proposition 2.1. *Let $x \in (0,1) \setminus \mathbb{Q}$ and $k \in \mathbb{N}$. We have:*

(1) $\mathrm{F}_{k+1} \leq q_k$*, where* F_j *is the* j*th Fibonacci number;*

(2) $\sum_{j=0}^{\infty} \frac{1}{q_j} < \infty$*;*

(3) $\sum_{j=0}^{k} q_j \leq 3q_k$*;*

(4) $\frac{q_k}{2q_{k+1}} \leq T^k(x) \leq \frac{2q_k}{q_{k+1}}$.

Proof. These properties can be deduced from the definitions. However, for the convenience of the reader we present the details. By definition $q_k = a_k q_{k-1} + q_{k-2}$, since $a_n \geq 1$ it follows that $\mathrm{F}_{k+1} \leq q_k$. Then $\sum_{j=0}^{\infty} \frac{1}{q_j} \leq \sum_{j=0}^{\infty} \frac{1}{\mathrm{F}_{j+1}}$ which converges. Also, $\sum_{j=0}^{k} q_j = q_k + q_{k-1} + \sum_{j=0}^{k-2} q_j = q_k + q_{k-1} + \sum_{j=0}^{k-2}(q_{j+2} - a_{j+2}q_{j+1}) \leq q_k + q_{k-1} + \sum_{j=0}^{k-2}(q_{j+2} - q_{j+1}) = q_k + q_{k-1} - q_1 + q_k \leq 3q_k$. Since $\left[\frac{1}{T^k(x)}\right] = a_{k+1}$, we have $a_{k+1} \leq \frac{1}{T^k(x)} \leq a_{k+1} + 1$. Finally, by definition of q_k, we have $a_{k+1} = \frac{q_{k+1} - q_{k-1}}{q_k}$, and hence $\frac{1}{T^k(x)} \leq \frac{q_{k+1} - q_{k-1}}{q_k} + 1 \leq \frac{2q_{k+1}}{q_k}$. On the other hand, $q_k = a_k q_{k-1} + q_{k-2} \geq 2q_{k-2}$ and $\frac{1}{T^k(x)} \geq \frac{q_{k+1} - q_{k-1}}{q_k} \geq \frac{q_{k+1} - q_{k+1}/2}{q_k} = \frac{q_{k+1}}{2q_k}$. This completes the proof of the Proposition. □

If $x \in \mathbb{Q}$, then $x = [a_0; a_1, a_2, ..., a_n]$ for some $n \in \mathbb{N}$. Proposition 2.1 holds also in this case for all p_k and q_k with $k \leq n$.

Let $\beta_k(x) = \prod_{j=0}^{k} T^j(x)$ for $k \geq 0$, and $\beta_{-1}(x) = 1$. Let $\gamma_k(x) = \beta_{k-1}(x)\log(\frac{1}{T^k(x)})$, for $k \geq 0$. Note that for all k and for all x,

$$0 \leq \beta_k(x) \leq 1 \text{ and } 0 \leq \gamma_k(x).$$

We state the important facts about $\beta_k(x)$ and $\gamma_k(x)$.

Proposition 2.2. *Let $x \in (0,1) \setminus \mathbb{Q}$. Then we have*

(1) $T^k(x) = -\frac{p_k - xq_k}{p_{k-1} - xq_{k-1}}$;

(2) $\beta_k(x) = (-1)^{k-1}(p_k - q_k x)$;

(3) $\beta_k(x) = \frac{1}{q_{k+1} + T^{k+1}(x) q_k}$, *for all* $k \geq -1$.

Proof. This follows from the definitions, however, for the convenience of the reader we present the details. We note that $x = [0; a_1, a_2, ..., a_k + T^k(x)] = -\frac{p_k + p_{k-1}T^k(x)}{q_k + q_{k-1}T^k(x)}$ and it follows that $T^k(x) = -\frac{p_k - xq_k}{p_{k-1} - xq_{k-1}}$ proving (1). Since $p_{-1} = 1$ and $q_{-1} = 0$, we get (2) by definition of β_k and (1). Substituting $x = -\frac{p_{k+1} + p_k T^{k+1}(x)}{q_{k+1} + q_k T^{k+1}(x)}$ gives (3). □

We now estimate the values of $\beta_k(x)$ and $\gamma_k(x)$.

Proposition 2.3. *Let $x \in (0,1) \setminus \mathbb{Q}$. We have:*

(1) $\frac{1}{2q_{k+1}} \leq \frac{1}{q_k + q_{k+1}} \leq \beta_k(x) \leq \frac{1}{q_{k+1}}$, *for all* $k \geq -1$;

(2) $\frac{\log(q_{k+1})}{q_k} - \frac{\log(2q_k)}{q_k} \leq \gamma_k(x) \leq \frac{\log(q_{k+1})}{q_k} + \frac{\log(2)}{q_k}$, *for all* $k \geq 0$.

Proof. These properties were derived in [MMY97, Proposition 1.4(iii)] and [BM12, Section 3]. However, for the convenience of the reader we present the details. By Proposition 2.2 (3) we obtain (1). The right-hand-side inequality of (2) follows from Proposition 2.1 (4) and Proposition 2.2 (3). For the other inequality, we note that $\gamma_k(x) = \frac{\log\left(\frac{1}{T^k(x)}\right)}{q_k + q_{k-1}T^k(x)}$ is a decreasing function in $T^k(x)$, since $T^k(x) \leq \frac{1}{a_{k+1}}$, we have $\gamma_k(x) \geq \frac{a_{k+1}\log(a_{k+1})}{q_{k+1}}$. If $a_{k+1} = 1$, then (1) is satisfied. If $a_{k+1} \geq 2$, then $\frac{q_{k+1}}{q_k} \geq 2$ and by the properties of the logarithm we have $\gamma_k(x) \geq \frac{\frac{q_{k+1} - q_{k-1}}{q_k}\log\left(\frac{q_{k+1} - q_{k-1}}{q_k}\right)}{q_{k+1}} \geq \frac{\left(\frac{q_{k+1}}{q_k} - 1\right)\log\left(\frac{q_{k+1}}{q_k} - 1\right)}{q_{k+1}} \geq \frac{\frac{q_{k+1}}{q_k}\log\left(\frac{q_{k+1}}{2q_k}\right)}{q_{k+1}} = \frac{\log\left(\frac{q_{k+1}}{2q_k}\right)}{q_k}$ completing the proof of the proposition. □

We can relate q_n and q_{n-1} via κ_n.

Lemma 2.4. *Let $x \in (0,1) \setminus \mathbb{Q}$, then*

$$\frac{1}{q_{n-1}^{\kappa_{n-1}-1}} \leq \frac{1}{q_n} \leq \frac{2}{q_{n-1}^{\kappa_{n-1}-1}}.$$

Proof. We have $\left|x - \frac{p_{n-1}}{q_{n-1}}\right| \leq \left|\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n}\right| = \left|\frac{p_{n-1}q_n - p_n q_{n-1}}{q_{n-1}q_n}\right| = \frac{1}{q_{n-1}q_n}$, by (2.1.1). On the other hand $\left|x - \frac{p_{n-1}}{q_{n-1}}\right| \geq \frac{1}{2}\left|\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n}\right| = \frac{1}{2q_{n-1}q_n}$. The result follows by the definition of κ_{n-1}. □

We call an open interval defined by endpoints $[0; a_1, a_2, ..., a_k]$ and $[0; a_1, a_2, ..., a_k + 1]$ a basic interval on the kth level $I(a_1, a_2, ..., a_k)$. The order depends on the parity of k. We will write $I_k(x)$ for the basic interval on the kth level that contains x. For all $x \in (0, 1) \setminus \mathbb{Q}$ and all $k \in \mathbb{N}$ there exists exactly one basic interval on the kth level that contains x. For $x \in (0, 1) \cap \mathbb{Q}$, we will say that x is of depth k if x belongs to some basic interval on the kth level but to none on the level $k + 1$. For $x \in (0, 1) \setminus \mathbb{Q}$ the end points of $I_k(x)$ are $\frac{p_k}{q_k}$ and $\frac{p_k + p_{k-1}}{q_k + q_{k-1}}$.

We now summarise some observations concerning the basic intervals.

Proposition 2.5. *Let $x \in (0, 1) \setminus \mathbb{Q}$. Then we have*

(1) the functions $T^i(x)$, $\beta_i(x)$, $\log(T^i(x))$, $\gamma_i(x)$ are continuous and differentiable on $I_k(x)$ for all $i \leq k$;

(2) for all $x \in I_k(x)$ we have $(T^k(x))' = \frac{(-1)^k}{\beta_{k-1}(x)^2}$;

(3) for all $x \in I_k(x)$ we have $\beta_{k-1}(x) = \frac{1}{q_k}(1 - q_{k-1}\beta_k(x))$.

Proof. The first statement follows from the definitions. Then differentiating the expression in Proposition 2.2 (1) we obtain $(T^k(x))' = -\frac{(-xq_k(p_{k-1} - xq_{k-1}) + (p_k - xq_k)q_{k-1}}{(p_{k-1} - xq_{k-1})^2}$. By (2.1.1) and Proposition 2.2 (2) we get (2), *cf.* [Riv12, Section 1.1]. Finally, it follows from (2.1.1) and Proposition 2.2 (2) that $1 = q_k\beta_{k-1}(x) + q_{k-1}\beta_k(x)$ and we obtain (3), *cf.* [MMY97, Proposition 1.4(iii)] . □

We can relate $\beta_k(x)$ to q_k using the following claim.

Claim 2.6. We have $(-1)^k\beta_k(x)\sum_{j=0}^{k}(-1)^j\frac{T^j(x)}{\beta_j(x)^2} = q_k$, for all x and all k.

Proof. We proceed by induction. If $k = 0$, then by convention

$$(-1)^0\beta_0(x)\sum_{j=0}^{0}(-1)^j\frac{T^j(x)}{\beta_j(x)^2} = \beta_0(x)\frac{1}{\beta_0(x)\beta_{-1}(x)} = 1 = q_0.$$

Assume $(-1)^{k-1}\beta_{k-1}(x)\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2} = q_{k-1}$. By Proposition 2.5 (3) we have

$$\begin{aligned}(-1)^k\beta_k(x)\sum_{j=0}^{k}(-1)^j\frac{T^j(x)}{\beta_j(x)^2} &= (-1)^k\frac{1 - q_k\beta_{k-1}}{q_{k-1}}\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2} + \frac{(-1)^k\beta_k(x)(-1)^k}{\beta_k(x)\beta_{k-1}(x)}\\ &= (-1)^k\frac{1 - q_k\beta_{k-1}}{q_{k-1}}(-1)^{k-1}\frac{q_{k-1}}{\beta_{k-1}(x)} + \frac{1}{\beta_{k-1}(x)}\end{aligned}$$

$$= -\frac{1}{\beta_{k-1}(x)} + q_k + \frac{1}{\beta_{k-1}(x)} = q_k.$$

This completes the proof of the claim. □

We will now prove that if $\mu(x) < \infty$, then x is square-Brjuno satisfying $(*)$. Assume that $\mu(x) < \infty$, then $\kappa_n \leq M$ for all $n \in \mathbb{N}$ and some $M > 2$. By Lemma 2.4, we have $\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n^2} \leq \sum_{n=0}^{\infty} \frac{\log q_n^{\kappa_n - 1}}{q_n^2} \leq \sum_{n=0}^{\infty} \frac{\kappa_n - 1 \log q_n}{q_n^2} \leq M \sum_{n=0}^{\infty} \frac{\log q_n}{q_n^2}$, which converges. Similarly, $\lim_{n\to\infty} \frac{\log q_{n+4}}{q_n^2} \leq \lim_{n\to\infty} M^4 \frac{\log q_n}{q_n^2} = 0$.

We finish this section with demonstrating that the square-Brjuno condition and $(*)$ are independent. Define $x \in (0,1) \setminus \mathbb{Q}$ by its partial quotients: for all $n \in \mathbb{N}$ let $a_{n+1} = \left[q_n^{\sqrt{q_n}}\right]$. We then have $\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n^2} = \sum_{n=0}^{\infty} \frac{\log(a_{n+1}q_n + q_{n-1})}{q_n^2} \leq \sum_{n=0}^{\infty} \left(\frac{\log(2q_n)}{q_n^2} + \frac{\log a_{n+1}}{q_n^2}\right) \leq \sum_{n=0}^{\infty} \left(\frac{\log(2q_n)}{q_n^2} + \frac{\sqrt{q_n}\log q_n}{q_n^2}\right)$ which converges. On the other hand, $\frac{\log q_{n+4}}{q_n^2} \geq \frac{\log a_{n+4}}{q_n^2} \geq \frac{\log a_{n+3}}{q_n^{3/2}} \geq \log q_n \to \infty$ as $n \to \infty$. This shows that x is square-Brjuno, but does not satisfy $(*)$. Consider now x defined by its partial quotients: for all $k \in \mathbb{N}$ let $a_{4k+1} = 2^{[q_{4k}^2/k]}$ and $a_n = 1$ otherwise. We then have $\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n^2} \geq \sum_{k=0}^{\infty} \frac{\log q_{4k+1}}{q_{4k}^2} \geq \sum_{k=0}^{\infty} \frac{\log a_{4k+1}}{q_{4k}^2} \geq \frac{1}{2}\sum_{k=0}^{\infty} \frac{\log 2}{k}$ which diverges. On the other hand, $\frac{\log q_{n+4}(x)}{q_n(x)^2} \leq \frac{4\log 2}{q_n^2} + \frac{\log q_n}{q_n^2} + \frac{\log(a_{n+4}a_{n+3}a_{n+2}a_{n+1})}{q_n^2}$ and only one of $a_{n+4}, a_{n+3}, a_{n+2}, a_{n+1}$ is different than 1. We than have, $\frac{\log q_{n+4}(x)}{q_n(x)^2} \leq \frac{4\log 2}{q_n^2} + \frac{\log q_n}{q_n^2} + \frac{4^4 \log 2}{n+3} \to 0$ as $n \to \infty$. This shows that x is not square-Brjuno, but it does satisfy $(*)$.

2.2 Modular and quasi-modular forms

In this thesis, one family of objects studied are Eisenstein series. For any $k \geq 4$, E_k is a modular form of weight k, not a cusp form. Whereas, E_2 is quasi-modular, that is for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$, we have

$$E_2(z) = \frac{E_2(\gamma \cdot z)}{(cz+d)^2} - \frac{6}{i\pi}\frac{c}{(cz+d)}. \tag{2.2.1}$$

Another family of functions is related to θ function. We recall some facts about it now. The theta modular group Γ_θ is the set of fractional transformations defined by $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mod 2$, or $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod 2$.

The θ function is defined in the upper-half plane as follows:

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{i\pi n^2 z}.$$

If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ defines a transformation from Γ_θ, then θ satisfies the following equality. For all $z \in \mathbb{H}$ we have

$$\theta(z) = \rho_\gamma (cz+d)^{-1/2}\theta(\gamma \cdot z), \tag{2.2.2}$$

where ρ_γ is a constant dependent only on c and d. It is equal to $\rho_\gamma = e^{\pi i m(-d/c)/4}$, with $m(-d/c)$ an integer defined as:

$$\begin{aligned} m(\infty) &= 0, \\ m(0) &= 1, \\ m(-d/c) &= m(-d/c+2), \\ m(c/d) &= m(-d/c) - \operatorname{sign}(-d/c). \end{aligned}$$

For more details, see for example [CQ09, VII.].

Remark 2.7. We have the following.

$$\begin{aligned} &\text{If } q = 1 \mod 4, \text{ then } m \in \{1,5\} \text{ and } \rho_\gamma = \pm\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right). \\ &\text{If } q = 3 \mod 4, \text{ then } m \in \{3,7\} \text{ and } \rho_\gamma = \pm\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right). \\ &\text{If } p = 1 \mod 4, \text{ then } m \in \{0,4\} \text{ and } \rho_\gamma = \pm 1. \\ &\text{If } p = 3 \mod 4, \text{ then } m \in \{2,6\} \text{ and } \rho_\gamma = \pm i. \end{aligned}$$

Remark 2.8. If $\gamma_1, \gamma_2 \in \Gamma_\theta$ such that $\gamma_1 \cdot y = \gamma_2 \cdot y = \infty$ for some y, then $\gamma_1 = \tau \circ \gamma_2$, where τ is a translation over an even integer.

Chapter 3
Itatsu's method

The differentiability of $F_{k,k+1}(x) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^{k+1}} \sin(2\pi nx)$ depends on the continued fraction of x. Already in 1933, Wilton in his work [Wil33] proved that there is a connection between some series involving the divisor functions and continued fractions. In this paper (among other series) he considered the following two series

$$\sum_{n=1}^{\infty} \frac{\sigma_0(n)}{n} \cos 2\pi nx; \quad \sum_{n=1}^{\infty} \frac{\sigma_0(n)}{n} \sin 2\pi nx.$$

He showed that the convergence of these series at x depends on the diophantine properties of x. This kind of series were first introduced by Riemann and also studied by Chowla and Walfisz [CW35], see also [LMZ10]

The approach developed by Itatsu has been implemented by various mathematicians. Recently, Balazard and Martin used it in studying the differentiability of the function

$$A(x) = \int_0^{\infty} \{t\}\{xt\} \frac{1}{t^2} dt,$$

where $\{y\}$ is the fractional part of y. The function $A(x)$ is interesting, because the Riemann hypothesis can be reformulated in terms of $A(x)$ or more precisely, the Nyman and Beurling criterion can be rephrased in terms of $A(x)$. Consider the Hilbert space $\mathcal{H} = L^2(0, +\infty; t^{-2}dt)$. For $\alpha \in \mathbb{R}$, let $e_\alpha(t) = \left\{\frac{t}{\alpha}\right\}$, $t > 0$. The Nyman and Beurling criterion says that the Riemann hypothesis is equivalent to the indicator function χ of $[1, +\infty)$ being the limit of the linear combinations of e_α, $\alpha \geq 1$ in $\mathcal{H}$. We have that $\mathrm{dist}_{\mathcal{H}}(\chi, \mathrm{Vect}(e_{\alpha_1}, ..., e_{\alpha_n}))^2 = \frac{\mathrm{Gram}(e_{\alpha_1}, ..., e_{\alpha_n}, \chi)}{\mathrm{Gram}(e_{\alpha_1}, ..., e_{\alpha_n})}$, with $\mathrm{Gram}(u_1, ..., u_n) = \det(\langle u_i, u_j \rangle)_{1 \leq i,j \leq n}$. For all $\alpha > 1$, we have $\langle e_\alpha, \chi \rangle = \frac{\log(\alpha) + 1 - \gamma}{\alpha}$, where γ is the Euler constant, and for all $\alpha, \beta > 0$ we have $\langle e_\alpha, e_\beta \rangle = \frac{1}{\alpha} A\left(\frac{\alpha}{\beta}\right) = \frac{1}{\beta} A\left(\frac{\beta}{\alpha}\right)$. Therefore, in order to study the distance $\mathrm{dist}_{\mathcal{H}}(\chi, \mathrm{Vect}(e_{\alpha_1}, ..., e_{\alpha_n}))$, we could study the function A. For details, see [BDBLS05]. It has been shown by Báez-Duarte, Balazard, Landreau and Saias that for all $x > 0$ we have

$$A(x) = \frac{1}{2}\log(x) + C + \frac{1}{2\pi^2 x} \sum_{n=1}^{\infty} \frac{\sigma_0(n)}{n^2} \cos(2\pi nx) - \frac{x}{\pi^2} \int_x^{\infty} \sum_{n=1}^{\infty} \frac{\sigma_0(n)}{n^2} \cos(2\pi nt) dt,$$

where C is a constant. Balazard and Martin proved that A is differentiable at x if and only if $x > 0 \notin \mathbb{Q}$, and $\sum_{k=0}^{\infty} (-1)^k \frac{\log(q_{k+1}(x))}{q_k(x)}$ converges, where $q_i(x)$ is the denominator of the ith convergent of x, see [BM12, BM13].

The first two sections of this chapter correspond to the papers [Pet14] and [Pet].

3.1 Differentiability of $F_{k,k+1}$ and $G_{k,k+1}$

In order to prove Theorem 1.2, we will proceed as Itatsu in [Ita81]. For $k > 0$ even, consider the complex valued function

$$\varphi_k(t) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^{k+1}} e^{2\pi int},$$

whose imaginary part is $F_{k,k+1}$ and real part is $G_{k,k+1}$. We start by the case when $k = 2$ and then we consider a general case.

3.1.1 Functional equation for φ_2

We use the convention that $0 \cdot \infty = 0$ and throughout this chapter we will work with the principal branch $-\pi < \arg(z) \leq \pi$ of $z \in \mathbb{C}$. We have the following proposition.

Proposition 3.1. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ *with* $c \neq 0$ *and* $x \in \mathbb{R}$. *We have*

$$\begin{aligned}\varphi_2(x) = (cx+d)^4\varphi_2(\gamma \cdot x) - \frac{i\pi^3}{3c^3}(cx+d)\mathrm{Log}(cx+d) + P_{-\frac{d}{c}}(x) \\ - \frac{\pi^2}{c^2}(cx+d)^2\mathrm{Log}(cx+d) + 6\int_{-\frac{d}{c}}^{x} c(ct+d)^2(c(x-t)-(ct+d))\varphi_2(\gamma \cdot t)dt,\end{aligned}$$

where Log *denotes the principal value of the complex logarithm and* $P_{-\frac{d}{c}}(x) \in \mathbb{C}[x]$ *is a polynomial of degree less than or equal to 3 that depends on* $\frac{c}{d}$.

The proof of Proposition 3.1 is very technical, therefore we will split the calculations into various lemmas and claims. Firstly, we note that φ_2 is differentiable in the upper-half plane, thus we have the following.

Claim 3.2. Let $z \in \mathbb{H}$. We have

$$\varphi_2'(z) = 2i\pi \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n^2} e^{2i\pi nz}, \tag{3.1.1}$$

$$\varphi_2''(z) = -4\pi^2 \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} e^{2i\pi nz}, \tag{3.1.2}$$

$$\varphi_2'''(z) = -8i\pi^3 \sum_{n=1}^{\infty} \sigma_1(n) e^{2i\pi nz} = \frac{i\pi^3}{3}E_2(z) - \frac{i\pi^3}{3}. \tag{3.1.3}$$

We then find a functional equation for φ_2'', which will be useful later.

Lemma 3.3. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ with $c \neq 0$ and $\tau, \alpha \in \mathbb{H}$. We have*

$$\begin{aligned}\varphi_2''(\tau) = \varphi_2''(\gamma\cdot\tau) - \varphi_2''(\gamma\cdot\alpha) - \frac{i\pi^3}{3c(c\tau+d)} + \frac{i\pi^3}{3c(c\alpha+d)} \\ - 2\pi^2\mathrm{Log}(c\tau+d) + 2\pi^2\mathrm{Log}(c\alpha+d) + \varphi_2''(\alpha) - \frac{i\pi^3}{3}\tau + \frac{i\pi^3}{3}\alpha.\end{aligned}$$

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ with $c \neq 0$ and $\tau, \alpha \in \mathbb{H}$. We have

$$\begin{aligned}\varphi_2''(\tau) &= -4\pi^2\left(\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}e^{2i\pi n\tau}\right) = \frac{i\pi^3}{3}\int_{i\infty}^{\tau}(E_2(t)-1)dt && \text{by (3.1.2)} \\ &= \frac{i\pi^3}{3}\int_{i\infty}^{\alpha}(E_2(t)-1)dt + \frac{i\pi^3}{3}\int_{\alpha}^{\tau}(E_2(t)-1)dt \\ &= \varphi_2''(\alpha) + \frac{i\pi^3}{3}\int_{\alpha}^{\tau}\left(\frac{E_2(\gamma\cdot t)}{(ct+d)^2} - \frac{6}{i\pi}\frac{c}{(ct+d)}\right)dt - \frac{i\pi^3}{3}\tau + \frac{i\pi^3}{3}\alpha && \text{by (2.2.1)} \\ &= \varphi_2''(\alpha) + \varphi_2''(\gamma\cdot\tau) - \varphi_2''(\gamma\cdot\alpha) - \frac{i\pi^3}{3c(c\tau+d)} + \frac{i\pi^3}{3c(c\alpha+d)} \\ &\quad - 2\pi^2\mathrm{Log}(c\tau+d) + 2\pi^2\mathrm{Log}(c\alpha+d) - \frac{i\pi^3}{3}\tau + \frac{i\pi^3}{3}\alpha,\end{aligned}$$

where Log denotes the principal value of the complex logarithm. □

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ with $c \neq 0$ and $z \in \mathbb{H}$ define

$$f_\gamma(z) = \varphi_2''(z) - \varphi_2''(\gamma\cdot z) + \frac{i\pi^3}{3c(cz+d)} + 2\pi^2\mathrm{Log}(cz+d) + \frac{i\pi^3}{3}z. \tag{3.1.4}$$

The next claim shows that f_γ depends only on c and d.

Claim 3.4. For each $\gamma \in SL_2(\mathbb{Z})$ the function f_γ is constant on $\mathbb{H}$. Moreover, if $\gamma_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c & d \end{smallmatrix}\right), \gamma_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, then $f_{\gamma_1} = f_{\gamma_2}$.

Proof. It follows from Lemma 3.3 that $f_\gamma(\tau) = f_\gamma(\alpha)$ for all $\tau, \alpha \in \mathbb{H}$, hence it must be a constant function on $\mathbb{H}$. Let $f_\gamma(z) = f_\gamma$ for all $z \in \mathbb{H}$. For the second part, let $\gamma_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c & d \end{smallmatrix}\right), \gamma_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Observe that the Lemma 3.3 implies that

$$\varphi_2''(\gamma_1\cdot z) - \varphi_2''(\gamma_2\cdot z) = f_{\gamma_1} - f_{\gamma_2},$$

for all $z \in \mathbb{H}$. Since f_γ does not depend on z, we have

$$\lim_{\substack{z\to -\frac{d}{c} \\ \mathrm{Im}(z)>0}} |\varphi_2''(\gamma_1\cdot z) - \varphi_2''(\gamma_2\cdot z)| = |f_{\gamma_1} - f_{\gamma_2}|.$$

Writing $z = x + iy$ we have

$$
\begin{aligned}
|\varphi_2''(\gamma_1 \cdot z) - \varphi_2''(\gamma_2 \cdot z)| &= 4\pi^2 \left| \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} e^{2i\pi n \gamma_1 \cdot z} - \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} e^{2i\pi n \gamma_2 \cdot z} \right| \\
&= 4\pi^2 \left| \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} \left(e^{2i\pi n \frac{a_1 x + b_1 + i a_1 y}{cx + d + icy}} - e^{2i\pi n \frac{a_2 x + b_2 + i a_1 y}{cx + d + icy}} \right) \right| \\
&\leq 4\pi^2 \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} e^{-\frac{2\pi n y}{(cx+d)^2 + (cy)^2}} \left| e^{2i\pi n \frac{a_1 c^2 + b_1 cx + a_1 dx + b_1 d + a_1 cy}{(cx+d)^2 + (cy)^2}} - e^{2i\pi n \frac{a_2 c^2 + b_2 cx + a_2 dx + b_2 d + a_1 cy}{(cx+d)^2 + (cy)^2}} \right| \\
&\leq 8\pi^2 \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} e^{-\frac{2\pi n y}{(cx+d)^2 + (cy)^2}}. \qquad (3.1.5)
\end{aligned}
$$

Since $x \to -\frac{d}{c}$ and $y \to 0^+$, as $z \to -\frac{d}{c}$, we conclude from (3.1.5) that

$$|\varphi_2''(\gamma_1 \cdot z) - \varphi_2''(\gamma_2 \cdot z)| \to 0$$

as $z \to -\frac{d}{c}$. This shows that $f_{\gamma_1} = f_{\gamma_2}$. □

We will now find a functional equation for φ_2'.

Lemma 3.5. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ with $c \neq 0$ and $\tau, \alpha \in \mathbb{H}$. We have*

$$
\begin{aligned}
\varphi_2'(\tau) = (c\tau + d)^2 \varphi_2'(\gamma \cdot \tau) - 2c(c\tau + d)^3 \varphi_2(\gamma \cdot \tau) + 6c^2 \int_{\alpha}^{\tau} (ct + d)^2 \varphi_2(\gamma \cdot t) dt \\
- \frac{i\pi^3}{3c^2} \mathrm{Log}(c\tau + d) - 2\pi^2 \frac{(c\tau + d)}{c} \mathrm{Log}(c\tau + d) + Q_{\gamma,\alpha}(\tau),
\end{aligned}
$$

where $Q_{\gamma,\alpha}(\tau) \in \mathbb{C}[\tau]$ of degree less than or equal to 2 depending on γ and α.

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ with $c \neq 0$ and $\tau, \alpha \in \mathbb{H}$. We have

$$
\begin{aligned}
\varphi_2'(\tau) &= \frac{i\pi^3}{3} \int_{i\infty}^{\tau} (\tau - t)(E_2(t) - 1) dt \qquad \text{by (3.1.1)} \\
&= \frac{i\pi^3}{3} \int_{i\infty}^{\alpha} (\tau - t)(E_2(t) - 1) dt + \frac{i\pi^3}{3} \int_{\alpha}^{\tau} (\tau - t) E_2(t) dt - \frac{i\pi^3}{3} \int_{\alpha}^{\tau} (\tau - t) dt \\
&= (\tau - \alpha)\varphi_2''(\alpha) + \varphi_2'(\alpha) + \frac{i\pi^3}{3} \int_{\alpha}^{\tau} (\tau - t) E_2(t) dt - \frac{i\pi^3}{6} (\tau - \alpha)^2. \qquad (3.1.6)
\end{aligned}
$$

We apply the relationship (2.2.1) and we integrate the remaining integral.

$$
\begin{aligned}
\int_{\alpha}^{\tau} (\tau - t) E_2(t) dt &= \int_{\alpha}^{\tau} (\tau - t) \Big(\frac{1}{(ct + d)^2} E_2(\gamma \cdot t) - \frac{6c}{i\pi(ct + d)} \Big) dt \\
&= \int_{\alpha}^{\tau} \frac{(\tau - t)}{(ct + d)^2} E_2(\gamma \cdot t) dt - \frac{6}{i\pi} \Big(\frac{(c\tau + d)}{c} \mathrm{Log}(c\tau + d) - \frac{(c\tau + d)}{c} \mathrm{Log}(c\alpha + d) - \tau + \alpha \Big)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{3}{i\pi^3}\int_\alpha^\tau \frac{(\tau-t)}{(ct+d)^2}\varphi_2'''(\gamma\cdot t)dt-\frac{1}{c^2}\mathrm{Log}(c\tau+d)+\frac{1}{c^2}\mathrm{Log}(c\alpha+d)+\frac{(c\tau+d)}{c^2(c\alpha+d)}-\frac{1}{c^2}\\
&-\frac{6}{i\pi}\Big(\frac{(c\tau+d)}{c}\mathrm{Log}(c\tau+d)-\frac{(c\tau+d)}{c}\mathrm{Log}(c\alpha+d)-\tau+\alpha\Big) \qquad \text{by (3.1.1)}\\
=&\frac{3}{i\pi^3}\Big(-(\tau-\alpha)\varphi_2''(\gamma\cdot\alpha)+(c\tau+d)^2\varphi_2'(\gamma\cdot\tau)-(c\alpha+d)^2\varphi_2'(\gamma\cdot\alpha)\\
&-2c(c\tau+d)^3\varphi_2(\gamma\cdot\tau)+3c(c\alpha+d)^3\varphi_2(\gamma\cdot\alpha)+6c^2\int_\alpha^\tau(ct+d)^2\varphi_2(\gamma\cdot t)dt\Big)\\
&-\frac{1}{c^2}\mathrm{Log}(c\tau+d)+\frac{1}{c^2}\mathrm{Log}(c\alpha+d)+\frac{(c\tau+d)}{c^2(c\alpha+d)}-\frac{1}{c^2}\\
&-\frac{6}{i\pi}\Big(\frac{(c\tau+d)}{c}\mathrm{Log}(c\tau+d)-\frac{(c\tau+d)}{c}\mathrm{Log}(c\alpha+d)-\tau+\alpha\Big).
\end{aligned}
$$

Substituting it into (3.1.6) gives

$$
\begin{aligned}
\varphi_2'(\tau)=(c\tau+d)^2\varphi_2'(\gamma\cdot\tau)-2c(c\tau+d)^3\varphi_2(\gamma\cdot\tau)+6c^2\int_\alpha^\tau(ct+d)^2\varphi_2(\gamma\cdot t)dt&\\
-\frac{i\pi^3}{3c^2}\mathrm{Log}(c\tau+d)-2\pi^2\frac{(c\tau+d)}{c}\mathrm{Log}(c\tau+d)+Q_{\gamma,\alpha}(\tau),&
\end{aligned}
$$

where $Q_{\gamma,\alpha}(\tau)=B'\tau^2+C'\tau+D'$, with

$$
\begin{aligned}
B'=&-\frac{i\pi^3}{6}\\
C'=&\varphi_2''(\alpha)-\varphi_2''(\gamma\cdot\alpha)+\frac{i\pi^3}{3c(c\alpha+d)}+2\pi^2\mathrm{Log}(c\alpha+d)+2\pi^2+\frac{i\pi^3}{3}\alpha\\
=&f_\gamma+2\pi^2 \qquad \text{by Lemma 3.3}
\end{aligned}
$$

$$
\begin{aligned}
D'=&-\alpha(\varphi_2''(\alpha)-\varphi_2''(\gamma\cdot\alpha))+\varphi_2'(\alpha)-(c\alpha+d)^2\varphi_2'(\gamma\cdot\alpha)+2c(c\alpha+d)^3\varphi_2(\gamma\cdot\alpha)\\
&+\frac{i\pi^3}{3c^2}\mathrm{Log}(c\alpha+d)+2\pi^2\frac{d}{c}\mathrm{Log}(c\alpha+d)+\frac{i\pi^3}{3}\frac{d}{c^2(c\alpha+d)}\\
&-\frac{i\pi^3}{3c^2}-2\pi^2\alpha-\frac{i\pi^3}{6}\alpha^2\\
=&-\alpha f_\gamma+\varphi_2'(\alpha)-(c\alpha+d)^2\varphi_2'(\gamma\cdot\alpha)+2c(c\alpha+d)^3\varphi_2(\gamma\cdot\alpha)\\
&+\frac{i\pi^3}{3c^2}\mathrm{Log}(c\alpha+d)+(c\alpha+d)\frac{2\pi^2}{c}\mathrm{Log}(c\alpha+d)-2\pi^2\alpha+\frac{i\pi^3}{6}\alpha^2 \qquad \text{by Lemma 3.3.}
\end{aligned}
$$

This completes the proof of the lemma. □

For $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$ with $c\neq 0$ and $\rho,z\in\mathbb{H}$ define

$$
g_\gamma(z,\rho)=-zf_\gamma+\varphi_2'(z)-(cz+d)^2\varphi_2'(\gamma\cdot z)+2c(ct+d)^3\varphi_2(\gamma\cdot z)+\frac{i\pi^3}{3c^2}\mathrm{Log}(cz+d)
$$

$$+ (cz+d)\frac{2\pi^2}{c}\mathrm{Log}(cz+d) - 2\pi^2 z + \frac{i\pi^3}{6}z^2 - 6c^2\int_\rho^z (ct+d)^2\varphi_2(\gamma\cdot t)dt.$$

The next claim shows that g_γ depends only on ρ and γ.

Claim 3.6. For each $\gamma \in SL_2(\mathbb{Z})$, for all $\rho \in \mathbb{H}$ we have $g_\gamma(z,\rho) = g_\gamma(w,\rho)$ for all $z, w \in \mathbb{H}$.

Proof. It follows from Lemma 3.5. □

For all $z \in \mathbb{H}$ write $g_\gamma(z,\rho) = g_\gamma(\rho)$. We note that Lemma 3.5 implies that

$$\varphi_2'(\alpha) - (c\alpha+d)^2\varphi_2'(\gamma\cdot\alpha) = g_\gamma(\alpha) + \alpha f_\gamma - 2c(c\alpha+d)^3\varphi_2(\gamma\cdot\alpha) - \frac{i\pi^3}{3c^2}\mathrm{Log}(c\alpha+d)$$
$$- (c\alpha+d)\frac{2\pi^2}{c}\mathrm{Log}(c\alpha+d) + 2\pi^2\alpha - \frac{i\pi^3}{6}\alpha^2. \quad (3.1.7)$$

We can now prove Proposition 3.1.
Proof of Proposition 3.1. Fix $\alpha \in \mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ with $c \neq 0$, let $\tau \in \mathbb{H}$. Integrating by parts we get

$$\begin{aligned}\varphi_2(\tau) =& \frac{i\pi^3}{6}\int_{i\infty}^\tau (\tau-t)^2(E_2(t)-1)dt\\ =& \frac{i\pi^3}{6}\int_\alpha^\tau (\tau-t)^2E_2(t)dt - \frac{i\pi^3}{6}\int_\alpha^\tau(\tau-t)^2dt + \frac{i\pi^3}{6}\int_{i\infty}^\alpha (\tau-t)^2(E_2(t)-1)dt\\ =& \frac{i\pi^3}{6}\int_\alpha^\tau (\tau-t)^2E_2(t)dt + \frac{i\pi^3(\alpha-\tau)^3}{18} + \frac{(\tau-\alpha)^2}{2}\varphi_2''(\alpha) + (\tau-\alpha)\varphi_2'(\alpha) + \varphi_2(\alpha).\end{aligned} \quad (3.1.8)$$

We apply (2.2.1) to the first term, we then obtain:

$$\begin{aligned}\frac{i\pi^3}{6}\int_\alpha^\tau (\tau-t)^2E_2(t)dt =& \frac{i\pi^3}{6}\int_\alpha^\tau (\tau-t)^2\left(\frac{1}{(ct+d)^2}E_2(\gamma\cdot t) - \frac{6c}{i\pi(ct+d)}\right)dt\\ =& \frac{i\pi^3}{6}\int_\alpha^\tau \frac{(\tau-t)^2}{(ct+d)^2}E_2(\gamma\cdot t)dt - \frac{\pi^2}{c^2}\Big((c\tau+d)^2\mathrm{Log}(c\tau+d) - (c\tau+d)^2\mathrm{Log}(c\alpha+d)\\ &- 2(c\tau+d)^2 + 2(c\tau+d)(c\alpha+d) + \frac{(c\tau+d)^2}{2} - \frac{(c\alpha+d)^2}{2}\Big).\end{aligned} \quad (3.1.9)$$

By (3.1.3), using the substitution $u = \gamma\cdot t$ and integrating by parts we get:

$$\begin{aligned}\frac{i\pi^3}{6}\int_\alpha^\tau \frac{(\tau-t)^2}{(ct+d)^2}E_2(\gamma\cdot t)dt =& \frac{1}{2}\int_\alpha^\tau \frac{(\tau-t)^2}{(ct+d)^2}\varphi_2'''(\gamma\cdot t)dt + \frac{i\pi^3}{6}\int_\alpha^\tau \frac{(\tau-t)^2}{(ct+d)^2}dt\\ =& \frac{1}{2}\int_\alpha^\tau \frac{(\tau-t)^2}{(ct+d)^2}\varphi_2'''(\gamma\cdot t)dt - \frac{i\pi^3}{3c^3}(c\tau+d)\mathrm{Log}(c\tau+d)\\ &+ \frac{i\pi^3}{6c^3}\Big(\frac{(c\tau+d)^2}{(c\alpha+d)} + 2(c\tau+d)\mathrm{Log}(c\alpha+d) - (c\alpha+d)\Big)\end{aligned}$$

$$
\begin{aligned}
&= \int_\alpha^\tau (\tau - t)\varphi_2''(\gamma\cdot t)dt - \frac{1}{2}(\tau-\alpha)^2\varphi_2''(\gamma\cdot\alpha) - \frac{i\pi^3}{3c^3}(c\tau+d)\mathrm{Log}(c\tau+d)\\
&\quad + \frac{i\pi^3}{6c^3}\Big(\frac{(c\tau+d)^2}{(c\alpha+d)} + 2(c\tau+d)\mathrm{Log}(c\alpha+d) - (c\alpha+d)\Big)\\
&= -\int_\alpha^\tau (ct+d)(2c\tau - 3ct - d)\varphi_2'(\gamma\cdot t)dt - (c\alpha+d)^2(\tau-\alpha)\varphi_2'(\gamma\cdot\alpha)\\
&\quad - \frac{(\tau-\alpha)^2}{2}\varphi_2''(\gamma\cdot\alpha) - \frac{i\pi^3}{3c^3}(c\tau+d)\mathrm{Log}(c\tau+d)\\
&\quad + \frac{i\pi^3}{6c^3}\Big(\frac{(c\tau+d)^2}{(c\alpha+d)} + 2(c\tau+d)\mathrm{Log}(c\alpha+d) - (c\alpha+d)\Big)\\
&= (c\tau+d)^4\varphi_2(\gamma\cdot\tau) + 2(c\tau+d)(c\alpha+d)^3\varphi_2(\gamma\cdot\alpha) - 3(c\alpha+d)^4\varphi_2(\gamma\cdot\alpha)\\
&\quad + \int_\alpha^\tau c(ct+d)^2(6c(\tau-t) - 6(ct+d))\varphi_2(\gamma\cdot t)dt - (c\alpha+d)^2(\tau-\alpha)\varphi_2'(\gamma\cdot\alpha)\\
&\quad - \frac{(\tau-\alpha)^2}{2}\varphi_2''(\gamma\cdot\alpha) - \frac{i\pi^3}{3c^3}(c\tau+d)\mathrm{Log}(c\tau+d)\\
&\quad + \frac{i\pi^3}{6c^3}\Big(\frac{(c\tau+d)^2}{(c\alpha+d)} + 2(c\tau+d)\mathrm{Log}(c\alpha+d) - (c\alpha+d)\Big).
\end{aligned}
\tag{3.1.10}
$$

Substituting (3.1.9) and (3.1.10) into (3.1.8) and gathering the terms we get

$$
\begin{aligned}
\varphi_2(\tau) = (c\tau+d)^4\varphi_2(\gamma\cdot\tau) &- \frac{i\pi^3}{3c^3}(c\tau+d)\mathrm{Log}(c\tau+d) + P_{\alpha,\gamma}(\tau)\\
&- \frac{\pi^2}{c^2}(c\tau+d)^2\mathrm{Log}(c\tau+d) + 6\int_\alpha^\tau c(ct+d)^2(c(\tau-t) - (ct+d))\varphi_2(\gamma\cdot t)dt,
\end{aligned}
$$

with $P_{\alpha,\gamma}(\tau) = A\tau^3 + B\tau^2 + C\tau + D$, where

$$
\begin{aligned}
A &= -\frac{i\pi^3}{18}\\
B &= \frac{1}{2}(\varphi_2''(\alpha) - \varphi_2''(\gamma\cdot\alpha)) + \frac{i\pi^3}{6}\alpha + \pi^2\mathrm{Log}(c\alpha+d) + \frac{3\pi^2}{2} + \frac{i\pi^3}{6c(c\alpha+d)}\\
&= \frac{1}{2}f_\gamma + \frac{3\pi^2}{2} && \text{by Lemma 3.3}
\end{aligned}
$$

$$
\begin{aligned}
C = &-\alpha(\varphi_2''(\alpha) - \varphi_2''(\gamma\cdot\alpha)) + \varphi_2'(\alpha) - (c\alpha+d)^2\varphi_2'(\gamma\cdot\alpha) + 2c(c\alpha+d)^3\varphi_2(\gamma\cdot\alpha) + \frac{3\pi^2 d}{c}\\
&+ \frac{2\pi^2 d}{c}\mathrm{Log}(c\alpha+d) + \frac{i\pi^3}{3c^2}\mathrm{Log}(c\alpha+d) - \frac{i\pi^3}{6}\alpha^2 - \frac{2\pi^2}{c}(c\alpha+d) + \frac{i\pi^3 d}{3c^2(c\alpha+d)}\\
= &g_\gamma(\alpha) + \frac{i\pi^3}{3c^2} + \frac{\pi^2 d}{c} \qquad \text{by Lemma 3.3 and (3.1.7)}\\
D = &\frac{1}{2}\alpha^2(\varphi_2''(\alpha) - \varphi_2''(\gamma\cdot\alpha)) - \alpha(\varphi_2'(\alpha) - (c\alpha+d)^2\varphi_2'(\gamma\cdot\alpha)) - (c\alpha+d)^4\varphi_2(\gamma\cdot\alpha)
\end{aligned}
$$

$$
\begin{aligned}
&- 2c(c\alpha+d)^3\alpha\varphi_2(\gamma\cdot\alpha)) + \varphi_2(\alpha) + \frac{\pi^2 d^2}{c^2}\mathrm{Log}(c\alpha+d) + \frac{i\pi^3 d}{3c^3}\mathrm{Log}(c\alpha+d)\\
&+ \frac{i\pi^3}{18}\alpha^3 + \frac{3\pi^2 d^2}{2c^2} - \frac{2\pi^2 d}{c^2}(c\alpha+d) + \frac{\pi^2}{2c^2}(c\alpha+d)^2 + \frac{i\pi^3 d^2}{6c^3(c\alpha+d)} - \frac{i\pi^3}{6c^3}(c\alpha+d)\\
=& -\frac{1}{2}\alpha^2 f_\gamma - \alpha g_\gamma(\alpha) - (c\alpha+d)^4\varphi_2(\gamma\cdot\alpha) + \varphi_2(\alpha) + (c\alpha+d)\frac{i\pi^3}{3c^3}\mathrm{Log}(c\alpha+d)\\
&+ (c\alpha+d)^2\frac{\pi^2}{c^2}\mathrm{Log}(c\alpha+d) - 2\pi^2\alpha^2 + \frac{i\pi^3}{18}\alpha^3 + \frac{3\pi^2 d^2}{2c^2} - \frac{2\pi^2 d}{c^2}(c\alpha+d)\\
&+ \frac{\pi^2}{2c^2}(c\alpha+d)^2 - \alpha\frac{i\pi^3}{3c^2}. \qquad \text{by Lemma 3.3 and (3.1.7)}
\end{aligned}
$$

Then we observe that if we let $\alpha \to -\frac{d}{c}$, then A, B, C, D are well defined. Moreover, since $D = \varphi_2(-\frac{d}{c})$ we have $g_\gamma\left(-\frac{d}{c}\right) = \frac{\pi^2 d}{2c} + \frac{i\pi^3 d^2}{18c^2} - \frac{i\pi^3}{3c^2} + \frac{d}{2c}f_\gamma$. Therefore, we obtain $A = -\frac{i\pi^3}{18}, B = \frac{1}{2}f_\gamma + \frac{3\pi^2}{2}, C = \frac{d}{2c}f_\gamma + \frac{3\pi^2 d}{2c} + \frac{i\pi^3 d^2}{18c^2}, D = \varphi_2\left(-\frac{d}{c}\right)$. By Claim 3.4, we deduce that the polynomial $P_{-\frac{d}{c},\gamma}$ depends only on d and c. Write $P_{-\frac{d}{c},\gamma} = P_{-\frac{d}{c}}$. Hence we have

$$
\begin{aligned}
\varphi_2(\tau) = {}& (c\tau+d)^4\varphi_2(\gamma\cdot\tau) - \frac{i\pi^3}{3c^3}(c\tau+d)\mathrm{Log}(c\tau+d) + P_{-\frac{d}{c}}(\tau)\\
&- \frac{\pi^2}{c^2}(c\tau+d)^2\mathrm{Log}(c\tau+d) + 6\int_{-\frac{d}{c}}^{\tau} c(ct+d)^2(c(\tau-t)-(ct+d))\varphi_2(\gamma\cdot t)dt.
\end{aligned}
$$

Letting $\tau \to x \in \mathbb{R}$ gives the result. □

3.1.2 Proof of Theorem 1.2

Before we start proving Theorem 1.2, we rewrite the polynomial $P_{-\frac{d}{c}}$ as

$$
P_{-\frac{d}{c}}(x) = \tilde{A}(cx+d)^3 + \tilde{B}(cx+d)^2 + \tilde{C}(cx+d) + \tilde{D}
$$

with

$$
\tilde{A} = -\frac{i\pi^3}{18c^3};\ \tilde{B} = \frac{f_\gamma}{2c^2} + \frac{3\pi^2}{2c^2} + \frac{i\pi^3 d}{6c^3};\ \tilde{C} = -\frac{d}{2c^2}f_\gamma + \frac{3\pi^2 d}{2c^2} - \frac{d^2 i\pi^3}{9c^3};\ \tilde{D} = \varphi_2\left(-\frac{d}{c}\right).
$$

Proof of Theorem 1.2. Let $\frac{p}{q} \in \mathbb{Q}$, p, q coprime, if $x = 0$, then let $q = 1$, $p = 0$. By Bézout's identity, we can choose $\gamma = \left(\begin{smallmatrix} a & b \\ q & -p \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. By Proposition 3.1 we have

$$
\begin{aligned}
\varphi_2(x) = {}& (qx-p)^4\varphi_2(\gamma\cdot x) - \frac{i\pi^3}{3q^3}(qx-p)\mathrm{Log}(qx-p) + P_{\frac{p}{q}}(x)\\
&- \frac{\pi^2}{q^2}(qx-p)^2\mathrm{Log}(qx-p) + 6\int_{\frac{p}{q}}^{x} q(qt-p)^2(q(x-t)-(qt-p))\varphi_2(\gamma\cdot t)dt.
\end{aligned}
$$

We observe that since $\varphi_2(x)$ is bounded on $\mathbb{R}$ we have

$$\left|6\int_{\frac{p}{q}}^{x} q(qt-p)^2(q(x-t)-(qt-p))\varphi_2(\gamma\cdot t)dt\right| \\ \leq c_1|q|\int_{\frac{p}{q}}^{x}(qt-p)^2|q(x-t)-(qt-p))|dt \leq c_2(qx-p)^4,$$

for some constants c_1, c_2.

As $x \to \frac{p}{q}^+$, Log becomes the natural logarithm log, and we have

$$\varphi_2(x) = \varphi_2\left(\frac{p}{q}\right) - \frac{i\pi^3}{3q^3}(qx-p)\log(qx-p) + \tilde{C}(qx-p) + O((qx-p)^2\log(qx-p)). \quad (3.1.11)$$

Taking the imaginary part of the both sides of Equation (3.1.11) shows that $F_{2,3}$ is not differentiable on $\frac{p}{q}$. On the other hand, taking the real part of the both sides of Equation (3.1.11) shows that $G_{2,3}$ is right-differentiable at $\frac{p}{q}$, and the value of the right derivative at $\frac{p}{q}$ is $q\mathrm{Re}(\tilde{C})$.

As $x \to \frac{p}{q}^-$, we have

$$\varphi_2(x) = \varphi_2\left(\frac{p}{q}\right) - \frac{i\pi^3}{3q^3}(qx-p)\log(|qx-p|) + \left(\tilde{C} + \frac{\pi^4}{3q^3}\right)(qx-p) + O((qx-p)^2\log(|qx-p|)), \quad (3.1.12)$$

where the coefficient $\frac{\pi^4}{3q^3}$ in front of $(qx-p)$ comes from the complex logarithm. Taking the real part of the both sides of Equation (3.1.12) shows that $G_{2,3}$ is also left-differentiable at $\frac{p}{q}$. The value of the left derivative at $\frac{p}{q}$ is $q\mathrm{Re}(\tilde{C}) + \frac{\pi^4}{3q^2}$. In particular, $G_{2,3}$ is not differentiable at $\frac{p}{q}$. At each rational $\frac{p}{q}$, if we denote the left and the right derivative of $G_{2,3}$ at $\frac{p}{q}$ by $G'_{2,3}\left(\frac{p}{q}^-\right), G'_{2,3}\left(\frac{p}{q}^+\right)$ respectively, then $G'_{2,3}\left(\frac{p}{q}^-\right) - G'_{2,3}\left(\frac{p}{q}^+\right) = \frac{\pi^4}{3q^2}$. This completes the proof of the Theorem. □

3.1.3 Functional equations for $F_{2,3}$ and $G_{2,3}$

We have the following proposition.

Proposition 3.7. *Let $x \in (0,1)$. We have*

$$F_{2,3}(x) = -x^4F_{2,3}(T(x)) - \frac{\pi^3}{3}x\log(x) + P(x) - 6\int_0^x t^2(x-2t)F_{2,3}(T(x))dt, \quad (3.1.13)$$

$$G_{2,3}(x) = x^4G_{2,3}(T(x)) - \pi^2x^2\log(x) + Q(x) + 6\int_0^x t^2(x-2t)G_{2,3}(T(x))dt, \quad (3.1.14)$$

where $P(x), Q(x) \in \mathbb{R}[x]$ are polynomials of degree less than or equal to 3.

We note that $\int_0^x t^2(x-2t)F_{2,3}(T(x))dt$ and $\int_0^x t^2(x-2t)G_{2,3}(T(x))dt$ are continuous and differentiable on $(0,1)$.

Proof. Let $x \in (0,1)$. We apply Proposition 3.1 with $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $x > 0$, $\mathrm{Log}(x)$ is just the natural logarithm $\log(x)$, and we obtain

$$\varphi_2(x) = x^4\varphi_2\Big(-\frac{1}{x}\Big) - \frac{i\pi^3}{3}x\log(x) + P_0(x) \\ - \pi^2x^2\log(x) + 6\int_0^x t^2(x-2t)\varphi_2\Big(-\frac{1}{x}\Big)dt, \quad (3.1.15)$$

where $P_0(x) = -\frac{i\pi^3}{18}x^3 + \left(\frac{f_\gamma}{2} + \frac{3\pi^2}{2}\right)x^2 + \varphi_2(0)$ with $f_\gamma(z) = 2i\pi^3$ obtained by evaluating (3.1.4) at $z = i$. We take imaginary and real parts of Equation (3.1.15) respectively and we get

$$F_{2,3}(x) = x^4F_{2,3}\Big(-\frac{1}{x}\Big) - \frac{\pi^3}{3}x\log(x) + \mathrm{Im}(P_0)(x) + 6\int_0^x t^2(x-2t)F_{2,3}\Big(-\frac{1}{x}\Big)dt,$$
$$G_{2,3}(x) = x^4G_{2,3}\Big(-\frac{1}{x}\Big) + \mathrm{Re}(P_0)(x) - \pi^2x^2\log(x) + 6\int_0^x t^2(x-2t)G_{2,3}\Big(-\frac{1}{x}\Big)dt.$$

Write $P = \mathrm{Im}(P_0)$, $Q = \mathrm{Re}(P_0)$. We conclude by observing that since $F_{2,3}$ is odd and $G_{2,3}$ is even, and they are both 1-periodic we have that $F_{2,3}\big(-\frac{1}{x}\big) = -F_{2,3}\big(\frac{1}{x}\big) = -F_{2,3}(T(x))$ and $G_{2,3}\big(-\frac{1}{x}\big) = G_{2,3}\big(\frac{1}{x}\big) = G_{2,3}(T(x))$. □

We iterate Equations (3.1.13) and (3.1.14) to obtain:

Corollary 3.8. *For all $n \in \mathbb{N}^*$ and $x \in (0,1)\setminus\mathbb{Q}$ we have:*

$$\begin{aligned} F_{2,3}(x) =& (-1)^nF_{2,3}(T^n(x))\beta_{n-1}(x)^4 + \frac{\pi^3}{3}\sum_{k=0}^{n}(-1)^k\beta_{k-1}(x)^2\beta_k(x)\gamma_k(x) \\ &+ \sum_{k=0}^{n}(-1)^kP(T^k(x))\beta_{k-1}(x)^4 \\ &+ 6\sum_{k=0}^{n}(-1)^{k+1}\beta_{k-1}(x)^4\int_0^{T^k(x)} t^2(T^k(x)-2t)F_{2,3}(T(t))dt, \end{aligned} \quad (3.1.16)$$

$$\begin{aligned} G_{2,3}(x) =& G_{2,3}(T^n(x))\beta_{n-1}(x)^4 + \pi^2\sum_{k=0}^{n}\beta_{k-1}(x)\beta_k(x)^2\gamma_k(x) \\ &+ \sum_{k=0}^{n}Q(T^k(x))\beta_{k-1}(x)^4 + 6\sum_{k=0}^{n}\beta_{k-1}(x)^4\int_0^{T^k(x)} t^2(T^k(x)-2t)G_{2,3}(T(t))dt. \end{aligned} \quad (3.1.17)$$

Letting $n \to \infty$, we get:

$$\begin{aligned}F_{2,3}(x) =&\frac{\pi^3}{3}\sum_{k=0}^{\infty}(-1)^k\beta_{k-1}(x)^2\beta_k(x)\gamma_k(x) + \sum_{k=0}^{\infty}(-1)^k P(T^k(x))\beta_{k-1}(x)^4 \\ &+ 6\sum_{k=0}^{\infty}(-1)^{k+1}\beta_{k-1}(x)^4\int_0^{T^k(x)} t^2(T^k(x)-2t)F_{2,3}(T(t))dt, \qquad (3.1.18)\\ G_{2,3}(x) =&\pi^2\sum_{k=0}^{\infty}\beta_{k-1}(x)\beta_k(x)^2\gamma_k(x) + \sum_{k=0}^{\infty}Q(T^k(x))\beta_{k-1}(x)^4 \\ &+ 6\sum_{k=1}^{\infty}\beta_{k-1}(x)^4\int_0^{T^k(x)} t^2(T^k(x)-2t)G_{2,3}(T(t))dt. \qquad (3.1.19)\end{aligned}$$

Proof. Equations (3.1.16) and (3.1.17) follow from iterating (3.1.13) and (3.1.14), respectively. Since $|F_{2,3}|$ and $|G_{2,3}|$ are bounded on $\mathbb{R}$, we have $|(-1)^n F_{2,3}(T^n(x))\beta_{n-1}(x)^4| \to 0$ and $|G_{2,3}(T^n(x))\beta_{n-1}(x)^4| \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned}F_{2,3}(x) = \sum_{k=0}^{\infty}\Big(&\frac{\pi^3}{3}(-1)^k\beta_{k-1}(x)^2\beta_k(x)\gamma_k(x) + (-1)^k P(T^k(x))\beta_{k-1}(x)^4 \\ &+ (-1)^{k+1}6\beta_{k-1}(x)^4\int_0^{T^k(x)} t^2(T^k(x)-2t)F_{2,3}(T(t))dt\Big), \qquad (3.1.20)\\ G_{2,3}(x) = \sum_{k=0}^{\infty}\Big(&\pi^2\beta_{k-1}(x)\beta_k(x)^2\gamma_k(x) + Q(T^k(x))\beta_{k-1}(x)^4 \\ &+ 6\beta_{k-1}(x)^4\int_0^{T^k(x)} t^2(T^k(x)-2t)G_{2,3}(T(t))dt\Big). \qquad (3.1.21)\end{aligned}$$

Finally, we note that $\big|\int_0^{T^k(x)} t^2(T^k(x)-2t)F_{2,3}(T(t))dt\big|$, $|P(T^k(x))|$ are bounded on $[0,1]$, therefore we have

$$\begin{aligned}\sum_{k=0}^{\infty}\Big(&\Big|\frac{\pi^3}{3}(-1)^k\beta_{k-1}(x)^2\beta_k(x)\gamma_k(x)\Big| + \Big|(-1)^k P(T^k(x))\beta_{k-1}(x)^4\Big| \\ &+ \Big|(-1)^{k+1}6\beta_{k-1}(x)^4\int_0^{T^k(x)} t^2(T^k(x)-2t)F_{2,3}(T(t))dt\Big|\Big) \\ &\le\frac{\pi^3}{3}\sum_{k=0}^{\infty}\beta_{k-1}(x)^2\beta_k(x)\gamma_k(x) + c_1\sum_{k=0}^{\infty}\beta_{k-1}(x)^4 \\ &\le\frac{2\pi^3}{3}\sum_{k=0}^{\infty}\frac{\log(q_{k+1})}{q_k^3 q_{k+1}} + c_1\sum_{k=0}^{\infty}\frac{1}{q_k^4} \qquad \text{by Proposition 2.3 (1) and (2)} \\ &\le c_2\sum_{k=0}^{\infty}\frac{1}{q_k^3} \le c_2\sum_{k=1}^{\infty}\frac{1}{\mathrm{F}_k^3} \qquad \text{by Proposition 2.1 (1),}\end{aligned}$$

for some constants c_1 and c_2. This shows that the series (3.1.20) converges absolutely and we can change the order of summation obtaining (3.1.18). In a similar way, we can show that (3.1.21) converges absolutely and we have (3.1.19). This completes the proof of the corollary. □

3.1.4 Proof of Theorem 1.3 (i)

Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Since $F_{2,3}$ is 1-periodic, we can assume $x \in (0,1)$. For brevity, let

$$\begin{aligned} u_{1,k}(x) =&(-1)^k \beta_{k-1}(x)^2 \beta_k(x) \gamma_k(x) \\ u_{2,k}(x) =&(-1)^k P(T^k(x)) \beta_{k-1}(x)^4 \\ u_{3,k}(x) =&(-1)^{k+1} \beta_{k-1}(x)^4 \int_0^{T^k(x)} t^2 (T^k(x) - 2t) F_{2,3}(T(t)) dt. \end{aligned} \tag{3.1.22}$$

With this notation, we have

$$\begin{aligned} F_{2,3}(x) &= (-1)^n F_{2,3}(T^n(x)) \beta_{n-1}(x)^4 + \frac{\pi^3}{3} \sum_{k=0}^{n} u_{1,k}(x) + \sum_{k=0}^{n} u_{2,k}(x) + 6 \sum_{k=0}^{n} u_{3,k}(x) \\ &= \frac{\pi^3}{3} \sum_{k=0}^{\infty} u_{1,k}(x) + \sum_{k=0}^{\infty} u_{2,k}(x) + 6 \sum_{k=0}^{\infty} u_{3,k}(x). \end{aligned}$$

We are interested in the limit $\frac{F_{2,3}(x+h) - F_{2,3}(x)}{h}$ as $h \to 0$. For each h, let $K_h \in \mathbb{N}$ such that $x + h \in I_k(x)$ for all $k \le K_h$ and $x + h \notin I_{K_h+1}(x)$ (where $I_k(x)$ denotes the basic interval on the kth level that contains x). By Corollary 3.8 we have

$$\begin{aligned} &\frac{F_{2,3}(x+h) - F_{2,3}(x)}{h} \\ &= \frac{(-1)^{K_h-1} \left(F_{2,3}(T^{K_h-1}(x+h)) \beta_{K_h-2}(x+h)^4 - F_{2,3}(T^{K_h-1}(x)) \beta_{K_h-2}(x)^4 \right)}{h} \\ &\quad + \frac{\frac{\pi^3}{3} \sum_{k=0}^{K_h-1} (u_{1,k}(x+h) - u_{1,k}(x))}{h} + \frac{\sum_{k=0}^{K_h-1} (u_{2,k}(x+h) - u_{2,k}(x))}{h} \\ &\quad + \frac{6 \sum_{k=0}^{K_h-1} (u_{3,k}(x+h) - u_{3,k}(x))}{h}. \end{aligned} \tag{3.1.23}$$

We are considering $(K_h - 1)$th iterate from Corollary 3.8, because the underlined idea is to use the Mean Value Theorem to estimate the values of $\frac{u_{i,k}(x+h) - u_{i,k}(x)}{h}$, $i \in \{1, 2, 3\}$. The functions $u_{i,k}$ are continuous and differentiable on $I_k(x)$, therefore $\frac{u_{i,k}(x+h) - u_{i,k}(x)}{h} = u'_{i,k}(t_{i,k})$, $i \in \{1, 2, 3\}$ for all $k \le K_h$ for some $t_{i,k}$ between x and $x + h$. However, we exclude K_h, because in the derivative of $u_{1,k}$ the factor $\gamma_k(x) = \log(\frac{1}{T^k(x)}) \prod_{j=0}^{k-1} T^j(x)$ appears. This factor can be estimated by $\frac{\log(q_{k+1})}{q_k} + O(q_k^{-1/2})$ (see Proposition 2.3 (2)). Since $x + h \notin I_{K_h+1}(x)$, we cannot relate $q_{K_h+1}(t_{1,K_h+1})$ to $q_{K_h+1}(x)$.

We calculate the limit of (3.1.23) as $h \to 0$. The calculation is long (but not very difficult), therefore we split it into various lemmas, in which we consider each term separately. Before we do it, we make the following observation.

Lemma 3.9. *Let $x \in (0,1) \setminus \mathbb{Q}, |h| > 0$ and K_h defined as above, then*

$$\frac{1}{2q_{K_h+2}q_{K_h+3}} \leq |h| \leq \frac{2}{q_{K_h}^2}.$$

If $a_k = 1$ only for finitely many indices k, then there exists $h_0 > 0$ such that if $|h| \leq h_0$ we have

$$\frac{1}{2q_{K_h+1}q_{K_h+2}} \leq |h| \leq \frac{2}{q_{K_h}^2}. \tag{3.1.24}$$

Proof. Since $x + h \in I_{K_h}(x)$, $|h|$ must be smaller than or equal to the distance from x to one of the endpoints of $I_{K_h}(x)$, which are $\frac{p_{K_h}}{q_{K_h}}$ and $\frac{p_{K_h}+p_{K_h-1}}{q_{K_h}+q_{K_h-1}}$. We then have

$$\begin{aligned}
|h| \leq& \max\left(\left|x - \frac{p_{K_h}}{q_{K_h}}\right|, \left|x - \frac{p_{K_h} + p_{K_h-1}}{q_{K_h} + q_{K_h-1}}\right|\right)\\
=& \max\left(\frac{\beta_{K_h}(x)}{q_{K_h}}, \frac{\beta_{K_h+1}(x)}{q_{K_h+1}} + \frac{a_{K_h+1} - 1}{q_{K_h+1}(q_{K_h} + q_{K_h-1})}\right)\\
\leq& \max\left(\frac{1}{q_{K_h}q_{K_h+1}}, \frac{1}{q_{K_h+1}q_{K_h+2}} + \frac{1}{(q_{K_h} + q_{K_h-1})(q_{K_h} + q_{K_h-1})}\right) \quad \text{by Proposition 2.3 (1)}\\
\leq& \frac{2}{q_{K_h}^2}.
\end{aligned}$$

On the other hand, since $x + h \notin I_{K_h+1}(x)$, $|h|$ must be greater than the distance from x to the boundary of $I_{K_h+1}(x)$. By [BM12, Proposition 4], $|h| \geq \frac{1}{2q_{K_h+2}q_{K_h+3}}$. If $a_k = 1$ only for finitely many indices k, then there exists $h_0 > 0$ such that for all $|h| \leq h_0$, for all $k \geq K_h$ we have $a_k > 1$. Then the distance from x to the boundary of $I_{K_h+1}(x)$ is greater than or equal to $\frac{1}{2q_{K_h+1}q_{K_h+2}}$, by [BM12, Proposition 4]. □

Remark 3.10. We cannot improve the lower bound on $|h|$ without imposing further conditions on x. To illustrate it, we show that we do not even have (3.1.24) in a general case. Let x a square-Brjuno number such that it has infinitely many continued fraction quotients equal to 1 and infinitely many different than 1, then there exists a sequence $(h_{K_n})_n$ such that $h_{K_n} \to 0$, as $n \to \infty$, $x + h_{K_n} \in I_{K_n}(x)$, $x + h_{K_n} \notin I_{K_n+1}(x)$ and $|h_{K_n}| \leq \frac{1}{q_{K_n+2}q_{K_n+3}}$. Indeed, let K_n such that: (1) K_1 is smallest possible and $K_{n+1} > K_n$; (2) $a_{K_n+2} = 1$ and $a_{K_n+3} \neq 1$. Then let $|h_{K_n}| > 0$ such that $x + h_{K_n} = \frac{p_{K_n+1}+p_{K_n}}{q_{K_n+1}+q_{K_n}}$. We have that $|h_{K_n}| \to 0$ as $n \to \infty$; $x + h_{K_n} \in I_{K_n}(x)$, $x + h_{K_n} \notin I_{K_n+1}(x)$; and $|h_{K_n}| \leq \frac{1}{q_{K_n+2}q_{K_n+3}}$.

We consider the first term.

Lemma 3.11. *Let $x \in (0,1) \setminus \mathbb{Q}$ such that it satisfies (∗) or (∗∗), then*

$$\frac{F_{2,3}(T^{K_h-1}(x+h))\beta_{K_h-2}(x+h)^4 - F_{2,3}(T^{K_h-1}(x))\beta_{K_h-2}(x)^4}{h} \to 0,$$

as $h \to 0$.

Proof. We have the following

$$\begin{aligned}\frac{F_{2,3}(T^{K_h-1}(x+h))\beta_{K_h-2}(x+h)^4 - F_{2,3}(T^{K_h-1}(x))\beta_{K_h-2}(x)^4}{h}&\\ = \beta_{K_h-2}(x)^4\left(\frac{F_{2,3}(T^{K_h-1}(x+h)) - F_{2,3}(T^{K_h-1}(x))}{h}\right)&\\ + \left(\frac{\beta_{K_h-2}(x+h)^4 - \beta_{K_h-2}(x)^4}{h}F_{2,3}(T^{K_h-1}(x+h))\right).&\end{aligned} \tag{3.1.25}$$

We consider the first summand. We have

$$\begin{aligned}&\left|\frac{F_{2,3}(T^{K_h-1}(x+h)) - F_{2,3}(T^{K_h-1}(x))}{h}\right|\\ &= \frac{|\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n^3}(\sin(2\pi n T^{K_h-1}(x+h)) - \sin(2\pi n T^{K_h-1}(x)))|}{|h|}\\ &= \frac{2|\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n^3}(\sin((T^{K_h-1}(x+h) - T^{K_h-1}(x))\pi n)\cos((T^{K_h-1}(x+h) + T^{K_h-1}(x))\pi n))|}{|h|}.\end{aligned}$$

Let N be the smallest integer greater or equal to $\frac{1}{h^2}$, that is $N = \lceil\frac{1}{h^2}\rceil$, then we have

$$\begin{aligned}&\left|\frac{F_{2,3}(T^{K_h-1}(x+h)) - F_{2,3}(T^{K_h-1}(x))}{h}\right|\\ &\le 2\frac{\sum_{n=1}^{N}\frac{\sigma_1(n)}{n^3}|\sin((T^{K_h-1}(x+h) - T^{K_h-1}(x))\pi n)|}{|h|} + 2\frac{\sum_{n=N+1}^{\infty}\frac{\sigma_1(n)}{n^3}}{|h|}\\ &\le 2\pi\frac{\sum_{n=1}^{N}\frac{\sigma_1(n)}{n^2}|T^{K_h-1}(x+h) - T^{K_h-1}(x)|}{|h|} + 2\frac{\sum_{n=N+1}^{\infty}\frac{\sigma_1(n)}{n^3}}{|h|}\\ &\le 8\pi q_{K_h-1}^2\sum_{n=1}^{N}\frac{\sigma_1(n)}{n^2} + 2\frac{\sum_{n=N+1}^{\infty}\frac{\sigma_1(n)}{n^3}}{|h|}.\end{aligned} \tag{3.1.26}$$

The last line follows from the fact that T^{K_h-1} is continuous and differentiable on $I_{K_h}(x)$, and by the Mean Value Theorem $\frac{|T^{K_h-1}(x+h) - T^{K_h-1}(x)|}{|h|} = |(T^{K_h-1}(t))'|$ for some t between x and $x+h$. By Proposition 2.5 (2.d) we have that $(T^k(y))' = (-1)^k\beta_{k-1}(y)^{-2}$. By Proposition 2.3 (1) we conclude that $|(T^{K_h-1}(t))'| \le 4q_{K_h-1}^2$.

Consider $\sum_{n=1}^{N} \frac{\sigma_1(n)}{n^2}$, by Abel's summation formula

$$\begin{aligned}\sum_{n=1}^{N} \frac{\sigma_1(n)}{n^2} =&\sum_{n=1}^{N-1}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)\sum_{k=1}^{n}\sigma_1(k)+\frac{1}{N^2}\sum_{n=1}^{N}\sigma_1(n)\\ \leq&3\sum_{n=1}^{N-1}\frac{1}{n^3}\sum_{k=1}^{n}\sigma_1(k)+\frac{1}{N^2}\sum_{n=1}^{N}\sigma_1(n).\end{aligned}$$

By Theorem 3 in [Ten95, p. 40], there exists $c_1 > 0$ such that $\sum_{j=1}^{k}\sigma_1(j) \leq \frac{\pi^2}{12}k^2 + c_1 k\log k$ for all $k \in \mathbb{N}$, and we have

$$\begin{aligned}\sum_{n=1}^{N} \frac{\sigma_1(n)}{n^2} \leq&3\sum_{n=1}^{N-1}\frac{1}{n^3}\left(\frac{\pi^2}{12}n^2+c_1 n\log n\right)+\frac{1}{N^2}\left(\frac{\pi^2}{12}N^2+c_1 N\log N\right)\\ \leq&\frac{\pi^2}{4}\sum_{n=1}^{N-1}\frac{1}{n}+3c_1\sum_{n=1}^{N-1}\frac{\log n}{n^2}+\frac{\pi^2}{12}+c_1\frac{\log N}{N}\\ \leq&\left(\frac{\pi^2}{4}+3c_1\right)\sum_{n=1}^{N-1}\frac{1}{n}+\frac{\pi^2}{12}+c_1\frac{\log N}{N}\leq c_2\log N,\end{aligned}\tag{3.1.27}$$

for some constant $c_2 > 0$, as $\sum_{n=1}^{k}\frac{1}{k} \leq \log(k)+2$ for all $k \in \mathbb{N}$.

Consider $\sum_{n=1}^{N}\frac{\sigma_1(n)}{n^3}$. By [Ten95, p. 88], we have $\sigma_1(k) \leq c_3 k\log(\log(k))$ for some constant c_3 for all $k \in \mathbb{N}$. We have

$$\begin{aligned}\sum_{n=N+1}^{\infty}\frac{\sigma_1(n)}{n^3} \leq&c_3\sum_{n=N+1}^{\infty}\frac{\log(\log(n))}{n^2}\leq c_3\sum_{n=N+1}^{\infty}\frac{(\log(n))^{1/2}}{n^2}\leq c_3\sum_{n=N+1}^{\infty}\frac{1}{n^{7/4}}\\ \leq&\frac{c_3}{(N+1)^{7/4}}+c_3\int_{N+1}^{\infty}\frac{1}{x^{7/4}}dx=\frac{c_3}{(N+1)^{7/4}}+\frac{4c_3}{3(N+1)^{3/4}}\leq\frac{7c_3}{3N^{3/4}}.\end{aligned}\tag{3.1.28}$$

Assume $|h| < 1$. Substituting (3.1.27) and (3.1.28) into (3.1.26), we get

$$\begin{aligned}\left|\frac{F_{2,3}(T^{K_h-1}(x+h))-F_{2,3}(T^{K_h-1}(x))}{h}\right| &\leq 8\pi c_2 q_{K_h-1}^2\log N+\frac{14c_3}{3N^{3/4}|h|}\\ &\leq 8\pi c_2 q_{K_h-1}^2\log\left(\frac{2}{h}\right)+\frac{14}{3}|h|^{1/2},\end{aligned}$$

by the choice of N.

By Lemma 3.9 and Proposition 2.3 (1), we have

$$\left|\frac{F_{2,3}(T^{K_h-1}(x+h))-F_{2,3}(T^{K_h-1}(x))}{h}\right|\beta_{K_h-2}(x)^4$$

$$
\begin{aligned}
&\leq \frac{8\pi c_2}{q_{K_h-1}^2} \log(4q_{K_h+2}q_{K_h+3}) + \frac{14}{3q_{K_h-1}^4}|h|^{1/2} \\
&\leq c_3 \frac{\log(q_{K_h+3})}{q_{K_h-1}^2} + \frac{14}{3q_{K_h-1}^4}|h|^{1/2},
\end{aligned}
$$

for some constant $c_3 > 0$. If x satisfies $(*)$, it converges to 0 as $h \to 0$. If x satisfies $(**)$, then Lemma 3.9 and Proposition 2.3 (1) imply

$$
\left| \frac{F_{2,3}(T^{K_h-1}(x+h)) - F_{2,3}(T^{K_h-1}(x))}{h} \right| \beta_{K_h-2}(x)^4 \leq c_4 \frac{\log(q_{K_h+2})}{q_{K_h-1}^2} + \frac{14}{3q_{K_h-1}^4}|h|^{1/2},
$$

for some constant $c_4 > 0$, and it converges to 0.

Finally, we consider the second summand of (3.1.25). Since the function $\beta_{K_h-2}(y)^4$ is continuous and differentiable on $I_{K_h}(x)$, the Mean Value Theorem implies that for some t between x and $x+h$ we have

$$
\begin{aligned}
\frac{|\beta_{K_h-2}(x+h)^4 - \beta_{K_h-2}(x)^4|}{|h|} &= |(\beta_{K_h-2}(t)^4)'| \\
&= 4\beta_{K_h-2}(t)^3(-1)^{K_h-2}q_{K_h-2} && \text{by Proposition 2.2 (2)} \\
&\leq \frac{4}{q_{K_h-1}^2} && \text{by Proposition 2.3 (1).}
\end{aligned}
$$

Observing that $|F_{2,3}|$ is bounded, and writing $\|F_{2,3}\|_\infty = \sup_{y\in[0,1)} |F_{2,3}(y)|$, we obtain

$$
\left| \frac{\beta_{K_h-1}(x+h)^4 - \beta_{K_h-1}(x)^4}{h} F_{2,3}(T^{K_h}(x+h)) \right| \leq \frac{4\|F_{2,3}\|_\infty}{q_{K_h-1}^2},
$$

which converges to 0 as $h \to 0$ for all $x \in (0,1)\backslash\mathbb{Q}$. This completes the proof of Lemma 3.11. $\square$

Lemma 3.12. *Let $x \in (0,1) \setminus \mathbb{Q}$ be a square-Brjuno number, then*

$$
\frac{\sum_{k=0}^{K_h-1} (u_{1,k}(x+h) - u_{1,k}(x))}{h}
$$
$$
\to \sum_{k=0}^{\infty} \beta_{k-1}(x)\gamma_k(x) + 4\sum_{k=0}^{\infty} \left((-1)^k \beta_{k-1}(x)^2 \beta_k(x)\gamma_k(x) \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2} \right) - \sum_{k=0}^{\infty} \beta_{k-1}(x)^2
$$

as $h \to 0$.

Before we start proving Lemma 3.12, we will prove the following two lemmas, which we will use in proving Lemma 3.12.

Lemma 3.13. *Let $x \in (0,1) \setminus \mathbb{Q}$. The series $\sum_{k=0}^{\infty} \beta_{k-1}(x)\gamma_k(x)$ converges if and only if $\sum_{k=0}^{\infty} \frac{\log(q_{k+1})}{q_k^2}$ converges.*

Proof. Since $\sum_{k=0}^{\infty} \beta_{k-1}(x)\gamma_k(x)$ is positive, we have:

$$\sum_{k=0}^{\infty} \beta_{k-1}(x)\gamma_k(x) \leq \sum_{k=0}^{\infty} \frac{\log(2q_{k+1})}{q_k^2} \leq \sum_{k=0}^{\infty} \frac{\log(2)}{q_k^2} + \sum_{k=0}^{\infty} \frac{\log(q_{k+1})}{q_k^2},$$

where the first inequality follows from Proposition 2.3. Since $\sum_{k=0}^{\infty} \frac{\log(2)}{q_k^2}$ converges for all $x \in (0,1) \setminus \mathbb{Q}$, if $\sum_{k=0}^{\infty} \frac{\log(q_{k+1})}{q_k^2}$ converges, then $\sum_{k=0}^{\infty} \beta_{k-1}(x)\gamma_k(x)$ converges as well. For the converse note that:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\log(q_{k+1})}{q_k^2} &\leq \sum_{k=0}^{\infty} \frac{\gamma_k(x)}{q_k} + \sum_{k=0}^{\infty} \frac{\log(2q_k)}{q_k^2} && \text{by Proposition 2.3 (2)} \\
&= \sum_{k=0}^{\infty} \beta_{k-1}(x)\gamma_k(x) + \sum_{k=0}^{\infty} \beta_{k-1}(x)\gamma_k(x)T^k(x)\frac{q_{k-1}}{q_k} + \sum_{k=0}^{\infty} \frac{\log(2q_k)}{q_k^2} && \text{by Proposition 2.2 (3)} \\
&\leq 2\sum_{k=0}^{\infty} \beta_{k-1}(x)\gamma_k(x) + \sum_{k=0}^{\infty} \frac{\log(2q_k)}{q_k^2},
\end{aligned}$$

as $T^k(x)\frac{q_{k-1}}{q_k} \leq 1$. The sum $\sum_{k=0}^{\infty} \frac{\log(2q_k)}{q_k^2}$ converges for all x, which completes the proof of the lemma. □

Lemma 3.14. *Both series*

$$4\sum_{k=0}^{\infty} \left((-1)^k \beta_{k-1}(x)^2 \beta_k(x)\gamma_k(x) \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2}\right), \quad \sum_{k=0}^{\infty} \beta_{k-1}(x)^2$$

converge for all $x \in (0,1) \setminus \mathbb{Q}$.

Proof. By Claim 2.6, we have

$$\begin{aligned}
&\left| 4\sum_{k=0}^{\infty} \left((-1)^k \beta_{k-1}(x)^2 \beta_k(x)\gamma_k(x) \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2}\right)\right| + \sum_{k=0}^{\infty} \beta_{k-1}(x)^2 \\
&\qquad \leq 4\sum_{k=0}^{\infty} |\beta_{k-1}(x)\beta_k(x)\gamma_k(x)q_{k-1}| + \sum_{k=0}^{\infty} \beta_{k-1}(x)^2 \\
&\qquad \leq 4\sum_{k=0}^{\infty} \frac{q_{k-1}}{q_k q_{k+1}} \frac{\log(2q_{k+1})}{q_k} + \sum_{k=0}^{\infty} \frac{1}{q_k^2} && \text{by Proposition 2.3} \\
&\qquad \leq 9\sum_{k=0}^{\infty} \frac{1}{q_k},
\end{aligned}$$

which converges by Proposition 2.1 (2). □

Proof of Lemma 3.12. Let $x \in (0,1) \setminus \mathbb{Q}$ be square-Brjuno. By Proposition 2.2 (2), for all $k \le K_h - 1$ we have

$$\begin{aligned}
u_{1,k}(x+h)-u_{1,k}(x) &= (-1)^k \beta_{k-1}(x+h)^2\beta_k(x+h)\gamma_k(x+h) - \beta_{k-1}(x)^2\beta_k(x)\gamma_k(x) \\
&= (-1)^{k+1}\Big(((x+h)q_k - p_k)\log(T^k(x+h))(-(x+h)q_{k-1}+p_{k-1})^3 \\
&\qquad - (xq_k - p_k)\log(T^k(x))(-xq_{k-1}+p_{k-1})^3\Big) \\
&= (-1)^{k+1}\Big(\beta_k(x)\beta_{k-1}(x)^3(\log(T^k(x+h)) - \log(T^k(x)))\Big) \\
&\quad + (-1)^{k+1}\Big((-1)^k\beta_{k-1}(x)^2 h + 4(-1)^{k-1}\beta_{k-1}(x)^2\beta_k(x)q_{k-1}h \\
&\qquad + A_k h^2 + B_k h^3 + C_k h^4\Big)\log(T^k(x+h)),
\end{aligned}$$

with

$$\begin{aligned}
A_k &= -3\beta_{k-1}(x)^2 q_{k-1}q_k + 6\beta_{k-1}(x)\beta_k(x)q_{k-1} + 3(-1)^k\beta_{k-1}(x)^2\beta_k(x)q_{k-1}^2 \\
B_k &= (-1)^k(3\beta_{k-1}(x)q_{k-1}^2 q_k - 3\beta_k(x)q_{k-1}^2 - \beta_k(x)q_{k-1}^3) - 3\beta_{k-1}(x)\beta_k(x)q_{k-1}^3 \\
C_k &= -q_{k-1}^3 q_k.
\end{aligned} \tag{3.1.29}$$

Therefore, by Proposition 2.5 and Claim 2.6, we have

$$\begin{aligned}
&\Bigg| \frac{\sum_{k=0}^{K_h-1}(u_{1,k}(x+h)-u_{1,k}(x))}{h} - \sum_{k=0}^{\infty}\beta_{k-1}(x)\gamma_k(x) \\
&\qquad - 4\sum_{k=0}^{\infty}\Big((-1)^k\beta_{k-1}(x)^2\beta_k(x)\gamma_k(x)\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\Big) + \sum_{k=0}^{\infty}\beta_{k-1}(x)^2\Bigg| \\
&\qquad\quad \le \Big|\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2\log(T^k(x+h)) + \sum_{k=0}^{K_h-1}\beta_{k-1}(x)\gamma_k(x)\Big| \\
&\qquad + 4\Big|\sum_{k=0}^{K_h-1}(-1)^{k+1}\Big(\beta_{k-1}(x)^3\beta_k(x)\log(T^k(x+h))\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\Big) \\
&\qquad\quad - \sum_{k=0}^{K_h-1}\Big((-1)^k\beta_{k-1}(x)^2\beta_k(x)\gamma_k(x)\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\Big)\Big| \\
&\quad + \Big|\sum_{k=0}^{K_h-1}(-1)^{k+1}\frac{\beta_{k-1}(x)^3\beta_k(x)(\log(T^k(x+h)) - \log(T^k(x)))}{h} + \sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2\Big| \\
&\quad + \Big|\sum_{k=0}^{K_h-1}(-1)^{k+1}A_k h\log(T^k(x+h))\Big| + \Big|\sum_{k=0}^{K_h-1}(-1)^{k+1}B_k h^2\log(T^k(x+h))\Big| \\
&\qquad + \Big|\sum_{k=0}^{K_h-1}(-1)^{k+1}C_k h^3\log(T^k(x+h))\Big| + \Big|\sum_{k=K_h}^{\infty}\beta_{k-1}(x)\gamma_k(x)
\end{aligned}$$

$$+ 4\sum_{k=K_h}^{\infty}\Big((-1)^k\beta_{k-1}(x)^2\beta_k(x)\gamma_k(x)\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\Big) - \sum_{k=K_h}^{\infty}\beta_{k-1}(x)^2\Big|. \quad (3.1.30)$$

By Lemmas 3.13 and 3.14, the last term converges to 0 as $h \to 0$. We will now show that all the other terms also converge to 0.

We observe that by Proposition 2.5, for all $k \leq K_h$ the function T^k is non-zero, continuous, and differentiable on $I_k(x)$, hence $\log(T^k)$ is continuous, and differentiable on $I_k(x)$. Then for all $k \leq K_h - 1$ and $y \in I_k(x)$ we have $\log(T^k(y))' = \frac{(-1)^k}{T^k(y)\beta_{k-1}(y)^2}$. By the Mean Value Theorem $|\log(T^k(x+h)) - \log(T^k(x))| = |h|\frac{1}{T^k(t_k)}\frac{1}{\beta_{k-1}(t_k)^2}$, for some t_k between x and $x+h$. Since $t_k \in I_k(x)$, by Proposition 2.1 (4) and 2.3 (1), we have $\frac{1}{T^k(t_k)}\frac{1}{\beta_{k-1}(t_k)^2} \leq \frac{2q_{k+1}}{q_k}4q_k^2 = 8q_kq_{k+1}$ and $|\log(T^k(x+h)) - \log(T^k(x))| \leq 8q_kq_{k+1}|h|$. Thus,

$$\begin{aligned}
&\Big|\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2\log(T^k(x+h)) + \sum_{k=0}^{K_h-1}\beta_{k-1}(x)\gamma_k(x)\Big| \\
&\qquad = \Big|\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2(\log(T^k(x+h)) - \log(T^k(x)))\Big| \\
&\qquad \leq 8|h|\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2q_kq_{k+1} \leq 8|h|\sum_{k=0}^{K_h-1}\frac{q_{k+1}}{q_k} \leq 16\frac{1}{q_{K_h}}\sum_{k=0}^{K_h-1}\frac{1}{q_k},
\end{aligned}$$

by Lemma 3.9, which converges to 0 as $h \to 0$.

Using the same arguments and applying Claim 2.6, we obtain

$$\begin{aligned}
4\Big|&\sum_{k=0}^{K_h-1}(-1)^{k+1}\Big(\beta_{k-1}(x)^3\beta_k(x)\log(T^k(x))\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\Big) \\
&\qquad - \sum_{k=0}^{K_h-1}\Big((-1)^k\beta_{k-1}(x)^2\beta_k(x)\gamma_k(x)\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\Big)\Big| \\
&\leq 4\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2\beta_k(x)q_{k-1}|\log(T^k(x+h))q_{k-1} - \log(T^k(x))| \\
&\leq 32\sum_{k=0}^{K_h-1}\frac{q_{k-1}}{q_k}|h| \leq 64\frac{1}{q_{K_h}}\sum_{k=0}^{K_h-1}\frac{1}{q_k},
\end{aligned}$$

which converges to 0 as $h \to 0$.

By the Mean Value Theorem, we have

$$\Big|\sum_{k=0}^{K_h-1}(-1)^{k+1}\frac{\beta_{k-1}(x)^3\beta_k(x)(\log(T^k(x+h)) - \log(T^k(x)))}{h} + \sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2\Big|$$

$$=\Big|\sum_{k=0}^{K_h-1}\frac{\beta_{k-1}(x)^3\beta_k(x)}{\beta_{k-1}(t_k)\beta_k(t_k)}-\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2\Big|,$$

for some t_k between x and $x+h$. Also, $\beta_{k-1}(y)\beta_k(y)$ is continuous and differentiable on $I_k(x)$ with the derivative $(\beta_{k-1}(y)\beta_k(y))' = (-1)^k\beta_k(y)q_{k-1} + (-1)^{k-1}\beta_{k-1}(y)q_k$. By Proposition 2.3 (1) for all $y \in I_k(x)$ we have $|(\beta_{k-1}(y)\beta_k(y))'| \le 2$. Therefore, we have

$$\begin{aligned}
&\Big|\sum_{k=0}^{K_h-1}(-1)^{k+1}\frac{\beta_{k-1}(x)^3\beta_k(x)(\log(T^k(x+h))-\log(T^k(x)))}{h}+\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2\Big|\\
&\quad=\Big|\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2\frac{-\beta_{k-1}(x)\beta_k(x)+\beta_{k-1}(t_k)\beta_k(t_k)}{\beta_{k-1}(t_k)\beta_k(t_k)}\Big|\\
&\quad\le 2\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2\frac{|x-t_k|}{\beta_{k-1}(t_k)\beta_k(t_k)}\le 8\sum_{k=0}^{K_h-1}\frac{q_{k+1}|h|}{q_k}\text{ by Proposition 2.3 (1)}\\
&\quad\le 16\frac{1}{q_{K_h}}\sum_{k=0}^{K_h-1}\frac{1}{q_k},
\end{aligned}$$

by Lemma 3.9, which converges to 0 as $h \to 0$ by Proposition 2.1 (2).

By Proposition 2.3 (1), we have $|A_k| \le 3\frac{q_{k-1}}{q_k(x)} + 6\frac{q_{k-1}}{q_kq_{k+1}} + 3\frac{q_{k-1}^2}{q_k^2q_{k+1}} \le 12$. Also by Proposition 2.1 (4), $|\log(T^k(x+h))| \le \frac{2q_{k+1}}{q_k}$. Then by Lemma 3.9, we have

$$\left|\sum_{k=0}^{K_h-1}(-1)^{k+1}A_kh\log(T^k(x+h))\right| \le 24\sum_{k=0}^{K_h-1}\frac{q_{k+1}}{q_k}|h| \le \frac{48}{q_{K_h}}\sum_{k=0}^{\infty}\frac{1}{q_k},$$

which converges to 0 as $h \to 0$ by Proposition 2.1 (2). Similarly, $|B_k| \le 3q_{k-1}^2 + 3\frac{q_{k-1}^2}{q_{k+1}} + 3\frac{q_{k-1}^3}{q_kq_{k+1}} + \frac{q_{k-1}^3}{q_{k+1}} \le 10q_{k-1}^2$. We then have

$$\left|\sum_{k=0}^{K_h-1}(-1)^{k+1}B_kh^2\log(T^k(x+h))\right| \le 20\sum_{k=0}^{K_h-1}\frac{q_{k-1}^2q_{k+1}}{q_k}h^2 \le 80\frac{1}{q_{K_h}}\sum_{k=0}^{\infty}\frac{1}{q_k}.$$

which converges to 0 as $h \to 0$ by Proposition 2.1 (2). Finally,

$$\left|\sum_{k=0}^{K_h-1}(-1)^{k+1}C_kh^3\log(T^k(x+h))\right| \le \sum_{k=0}^{K_h-1}\frac{q_{k-1}^3q_kq_{k+1}}{q_k}|h|^3 \le 8\frac{1}{q_{K_h}}\sum_{k=0}^{\infty}\frac{1}{q_k},$$

which converges to 0 as $h \to 0$ by Proposition 2.1 (2).

This shows that (3.1.30) converges to 0 as $h \to 0$ completing the proof of the lemma.

□

Lemma 3.15. *Let $x \in (0,1) \setminus \mathbb{Q}$, then*

$$\frac{\sum_{k=0}^{K_h-1} (u_{2,k}(x+h) - u_{2,k}(x))}{h}$$
$$\to \sum_{k=0}^{\infty} (P(T^k(x)))' \beta_{k-1}(x)^2 + 4 \sum_{k=0}^{\infty} (-1)^k P(T^k(x)) \beta_{k-1}(x)^4 \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2},$$

as $h \to 0$, where $(P(T^k(x)))'$ is the derivative of the polynomial P evaluated at $T^k(x)$.

Before we start proving Lemma 3.15, we will prove the following lemma, which we will use in proving Lemma 3.15.

Lemma 3.16. *The series*

$$\sum_{k=0}^{\infty} \big((P(T^k(x)))' \beta_{k-1}(x)^2 + (-1)^k 4 P(T^k(x)) \beta_{k-1}(x)^4 \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2} \big)$$

converges for all $x \in (0,1) \setminus \mathbb{Q}$.

Proof. Firstly, by Claim 2.6 we have $q_{k-1} = (-1)^{k-1} \beta_{k-1}(x) \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2}$. Write $\|P\|_\infty = \sup_{y \in (0,1)} |P(y)|$ and $\|P'\|_\infty = \sup_{y \in (0,1)} |P(y)'|$. Since P and P' are polynomials, we have $\|P\|_\infty$ and $\|P'\|_\infty$ are finite. We then have:

$$\begin{aligned}
&\Big| \sum_{k=0}^{\infty} \big((P(T^k(x)))' \beta_{k-1}(x)^2 + (-1)^k 4 P(T^k(x)) \beta_{k-1}(x)^4 \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2} \big) \Big| \\
&\qquad \le \|P'\|_\infty \sum_{k=0}^{\infty} \beta_{k-1}(x)^2 + 4 \|P\|_\infty \sum_{k=0}^{\infty} \beta_{k-1}(x)^3 q_{k-1} \\
&\qquad \le \|P'\|_\infty \sum_{k=0}^{\infty} \frac{1}{q_{k-1}^2} + 4 \|P\|_\infty \sum_{k=0}^{\infty} \frac{q_{k-1}}{q_k^3} \quad \text{by Proposition 2.3 (1)} \\
&\qquad \le (\|P'\|_\infty + 4 \|P\|_\infty) \sum_{k=0}^{\infty} \frac{1}{q_{k-1}^2},
\end{aligned}$$

which converges for all $x \in (0,1) \setminus \mathbb{Q}$ by Proposition 2.1 (2). □

Proof of Lemma 3.15. Let $x \in (0,1) \setminus \mathbb{Q}$. We have $u_{2,k}(x) = (-1)^k P(T^k(x)) \beta_{k-1}(x)^4$ where $P(y) = \hat{A} y^3 + \hat{B} y^2 + \hat{C} y + \hat{D}$, for some constants $\hat{A}, \hat{B}, \hat{C}, \hat{D} \in \mathbb{R}$. We then have

$$\frac{\sum_{k=0}^{K_h-1} (u_{2,k}(x+h) - u_{2,k}(x))}{h}$$
$$= \frac{\hat{A}}{h} \sum_{k=0}^{K_h-1} (-1)^k \big(\beta_k(x+h)^3 \beta_{k-1}(x+h) - \beta_k(x)^3 \beta_{k-1}(x) \big)$$

$$
\begin{aligned}
&+\frac{\hat{B}}{h}\sum_{k=0}^{K_h-1}(-1)^k\left(\beta_k(x+h)^2\beta_{k-1}(x+h)^2-\beta_k(x)^2\beta_{k-1}(x)^2\right)\\
&+\frac{\hat{C}}{h}\sum_{k=0}^{K_h-1}(-1)^k\left(\beta_k(x+h)\beta_{k-1}(x+h)^3-\beta_k(x)\beta_{k-1}(x)^3\right)\\
&\qquad+\frac{\hat{D}}{h}\sum_{k=0}^{K_h-1}(-1)^k\left(\beta_{k-1}(x+h)^4-\beta_{k-1}(x)^4\right). \quad (3.1.31)
\end{aligned}
$$

We consider each term separately. By Proposition 2.2 (2), for all $k \le K_h - 1$ we have

$$
\begin{aligned}
&(-1)^k\left(\beta_{k-1}(x+h)\beta_k(x+h)^3-\beta_{k-1}(x)\beta_k(x)^3\right)\\
&\quad=(-1)^k\left(((-1)^{k-1}(p_k-xq_k-hq_k))^3(-1)^k(p_{k-1}-xq_{k-1}-hq_{k-1})-\beta_k(x)^3\beta_{k-1}(x)\right)\\
&\quad=(-1)^k\left((\beta_k(x)+(-1)^khq_k)^3(\beta_{k-1}(x)+(-1)^{k-1}hq_{k-1})-\beta_k(x)^3\beta_{k-1}(x)\right)\\
&\quad=3h\beta_{k-1}(x)\beta_k(x)^2q_k-h\beta_k(x)^3q_{k-1}+3h^2(-1)^k\beta_{k-1}(x)\beta_k(x)q_k^2+h^3\beta_{k-1}(x)q_k^3\\
&\qquad\quad+3h^2(-1)^{k-1}\beta_k(x)^2q_{k-1}q_k-3h^3\beta_k(x)q_{k-1}q_k^2+h^4(-1)^{k-1}q_{k-1}q_k^3.
\end{aligned}
$$

Let

$$
\begin{aligned}
S_1(h)=3h\sum_{k=0}^{K_h-1}(-1)^k\beta_{k-1}(x)\beta_k(x)q_k^2+h^2\sum_{k=0}^{K_h-1}\beta_{k-1}(x)q_k^3+3h\sum_{k=0}^{K_h-1}(-1)^{k-1}\beta_k(x)^2q_{k-1}q_k\\
-3h^2\sum_{k=0}^{K_h-1}\beta_k(x)q_{k-1}q_k^2+h^3\sum_{k=0}^{K_h-1}(-1)^{k-1}q_{k-1}q_k^3.
\end{aligned}
$$

We now consider the second term of (3.1.31). Again by Proposition 2.2 (2), for all $k \le K_h - 1$ we have

$$
\begin{aligned}
&(-1)^k\left(\beta_k(x+h)^2\beta_{k-1}(x+h)^2-\beta_k(x)^2\beta_{k-1}(x)^2\right)\\
&\quad=(-1)^k\left(((-1)^{k-1}(p_k-xq_k-hq_k))^2((-1)^k(p_{k-1}-xq_{k-1}-hq_{k-1}))^2-\beta_k(x)^2\beta_{k-1}(x)^2\right)\\
&\quad=(-1)^k\left((\beta_k(x)+(-1)^khq_k)^2(\beta_{k-1}(x)+(-1)^{k-1}hq_{k-1})^2-\beta_k(x)^2\beta_{k-1}(x)^2\right)\\
&\quad=2h\beta_{k-1}(x)^2\beta_k(x)q_k-2h\beta_{k-1}(x)\beta_k(x)^2q_{k-1}+h^2(-1)^k\beta_{k-1}(x)^2q_k^2\\
&\qquad-4h^2(-1)^k\beta_{k-1}(x)\beta_k(x)q_{k-1}q_k-2h^3\beta_{k-1}(x)q_{k-1}q_k^2+h^2(-1)^k\beta_k(x)^2q_{k-1}^2\\
&\qquad+2h^3\beta_k(x)q_{k-1}^2q_k+h^4(-1)^kq_{k-1}^2q_k^2.
\end{aligned}
$$

Let

$$
\begin{aligned}
S_2(h)=h\sum_{k=0}^{K_h-1}(-1)^k\beta_{k-1}(x)^2q_k^2-4h\sum_{k=0}^{K_h-1}(-1)^k\beta_{k-1}(x)\beta_k(x)q_{k-1}q_k\\
-2h^2\sum_{k=0}^{K_h-1}\beta_{k-1}(x)q_{k-1}q_k^2+h\sum_{k=0}^{K_h-1}(-1)^k\beta_k(x)^2q_{k-1}^2
\end{aligned}
$$

$$+ 2h^2 \sum_{k=0}^{K_h-1} \beta_k(x) q_{k-1}^2 q_k + h^3 \sum_{k=0}^{K_h-1} (-1)^k q_{k-1}^2 q_k^2.$$

We now consider the third term of (3.1.31). Similarly, by Proposition 2.2 (2), for all $k \le K_h - 1$ we have

$$\begin{aligned}
&(-1)^k \left(\beta_k(x+h)\beta_{k-1}(x+h)^3 - \beta_k(x)\beta_{k-1}(x)^3\right) \\
&\quad = (-1)^k \left((-1)^{k-1}(p_k - xq_k - hq_k)((-1)^k(p_{k-1} - xq_{k-1} - hq_{k-1}))^3 - \beta_k(x)\beta_{k-1}(x)^3\right) \\
&\quad = (-1)^k \left((\beta_k(x) + (-1)^k hq_k)(\beta_{k-1}(x) + (-1)^{k-1} hq_{k-1})^3 - \beta_k(x)\beta_{k-1}(x)^3\right) \\
&\quad = -3h\beta_{k-1}(x)^2\beta_k(x)q_{k-1} + h\beta_{k-1}(x)^3 q_k + 3h^2(-1)^k \beta_{k-1}(x)\beta_k(x)q_{k-1}^2 \\
&\qquad - h^3\beta_k(x)q_{k-1}^3 + 3h^2(-1)^{k-1}\beta_{k-1}(x)^2 q_{k-1}q_k + 3h^3\beta_{k-1}(x)q_{k-1}^2 q_k \\
&\qquad + h^4(-1)^{k-1}q_{k-1}^3 q_k.
\end{aligned}$$

Let

$$\begin{aligned}
S_3(h) = 3h \sum_{k=0}^{K_h-1} (-1)^k \beta_{k-1}(x)\beta_k(x)q_{k-1}^2 - h^2 \sum_{k=0}^{K_h-1} \beta_k(x)q_{k-1}^3 \\
+ 3h \sum_{k=0}^{K_h-1} (-1)^{k-1}\beta_{k-1}(x)^2 q_{k-1}q_k + 3h^2 \sum_{k=0}^{K_h-1} \beta_{k-1}(x)q_{k-1}^2 q_k \\
+ h^3 \sum_{k=0}^{K_h-1} (-1)^{k-1} q_{k-1}^3 q_k.
\end{aligned}$$

Finally, we consider the last term of (3.1.31). As before, by Proposition 2.2 (2), for all $k \le K_h - 1$ we have

$$\begin{aligned}
&(-1)^k \left(\beta_{k-1}(x+h)^4 - \beta_{k-1}(x)^4\right) \\
&\quad = (-1)^k \left(((-1)^k(p_{k-1} - xq_{k-1} - hq_{k-1}))^4 - \beta_{k-1}(x)^4\right) \\
&\quad = (-1)^k \left((\beta_{k-1}(x) + (-1)^{k-1}hq_{k-1}))^4 - \beta_{k-1}(x)^4\right) \\
&\quad = -4h\beta_{k-1}(x)^3 q_{k-1} + 6h^2(-1)^k\beta_{k-1}(x)^2 q_{k-1}^2 - 4h^3\beta_{k-1}(x)q_{k-1}^3 + h^4(-1)^k q_{k-1}^4.
\end{aligned}$$

Let

$$S_4(h) = 6h \sum_{k=0}^{K_h-1} (-1)^k \beta_{k-1}(x)^2 q_{k-1}^2 - 4h^2 \sum_{k=0}^{K_h-1} \beta_{k-1}(x) q_{k-1}^3 + h^3 \sum_{k=0}^{K_h-1} (-1)^k q_{k-1}^4.$$

Then we have

$$\sum_{k=0}^{K_h-1} \frac{(u_{3,k}(x+h) - u_{3,k}(x))}{h}$$

$$
\begin{aligned}
&= \hat{A}(3\sum_{k=0}^{K_h-1}\beta_{k-1}(x)\beta_k(x)^2q_k - \sum_{k=0}^{K_h-1}\beta_k(x)^3q_{k-1} + S_1(h)) \\
&\quad + \hat{B}(2\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2\beta_k(x)q_k - 2\sum_{k=0}^{K_h-1}\beta_{k-1}(x)\beta_k(x)^2q_{k-1} + S_2(h)) \\
&\quad + \hat{C}(-3\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^2\beta_k(x)q_{k-1} + \sum_{k=0}^{K_h-1}\beta_{k-1}(x)^3q_k + S_3(h)) \\
&\quad + \hat{D}(-4\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^3q_{k-1} + S_4(h)) \\
&= \sum_{k=0}^{K_h-1}3\hat{A}\beta_{k-1}(x)^3(T^k(x))^2q_k - \sum_{k=0}^{K_h-1}\hat{A}(T^k(x)^3)\beta_{k-1}(x)^3q_{k-1} + \hat{A}S_1(h) \\
&\quad + \sum_{k=0}^{K_h-1}2\hat{B}\beta_{k-1}(x)^2T^k(x)q_k - 2\sum_{k=0}^{K_h-1}\hat{B}\beta_{k-1}(x)\beta_k(x)^3q_{k-1} + \hat{B}S_2(h) \\
&\quad + \sum_{k=0}^{K_h-1}\hat{C}\beta_{k-1}(x)^3q_k - 3\sum_{k=0}^{K_h-1}\hat{C}\beta_{k-1}(x)^3T^k(x)q_{k-1} + \hat{C}S_3(h) \\
&\quad - 4\hat{D}\sum_{k=0}^{K_h-1}\beta_{k-1}(x)^3q_{k-1} + \hat{D}S_4(h) \\
&= \sum_{k=0}^{K_h-1}(3\hat{A}(T^k(x))^2 + 2\hat{B}T^k(x) + \hat{C})\beta_{k-1}(x)^3q_k \\
&\quad - \sum_{k=0}^{K_h-1}(\hat{A}(T^k(x)^3) + 2\hat{B}(T^k(x)^2) + 3\hat{C}T^k(x) + 4\hat{D})\beta_{k-1}(x)^3q_{k-1} \\
&\quad + \hat{A}S_1 + \hat{B}S_2 + \hat{C}S_3 + \hat{D}S_4.
\end{aligned}
$$

By Proposition 2.2 (3), we have

$$
\begin{aligned}
&(3\hat{A}(T^k(x))^2 + 2\hat{B}T^k(x) + \hat{C})\beta_{k-1}(x)^3q_k \\
&\quad = (3\hat{A}(T^k(x))^2 + 2\hat{B}T^k(x) + \hat{C})\beta_{k-1}(x)^3\frac{1}{\beta_{k-1}(x)}(1 - q_{k-1}\beta_k(x)) \\
&\quad = (3\hat{A}(T^k(x))^2 + 2\hat{B}T^k(x) + \hat{C})\beta_{k-1}(x)^2(1 - q_{k-1}\beta_k(x)) \\
&\quad = (3\hat{A}(T^k(x))^2 + 2\hat{B}T^k(x) + \hat{C})\beta_{k-1}(x)^2 \\
&\qquad - (3\hat{A}(T^k(x))^3 + 2\hat{B}(T^k(x))^2 + \hat{C}T^k(x))\beta_{k-1}(x)^3q_{k-1}.
\end{aligned}
$$

Hence, we have

$$
\frac{\sum_{k=0}^{K_h-1}(u_{2,k}(x+h) - u_{2,k}(x))}{h}
$$

$$= \sum_{k=0}^{K_h-1} (P(T^k(x)))'\beta_{k-1}(x)^2 - 4\sum_{k=0}^{K_h-1} P(T^k(x))\beta_{k-1}(x)^3 q_{k-1} \\ + \hat{A}S_1(h) + \hat{B}S_2(h) + \hat{C}S_3(h) + \hat{D}S_4(h),$$

where $(P(T^k(x)))'$ is the derivative of the polynomial P evaluated at $T^k(x)$, that is

$$(P(T^k(x)))' = 3\hat{A}(T^k(x))^2 + 2\hat{B}T^k(x) + \hat{C}.$$

We then have

$$\Big|\frac{\sum_{k=0}^{K_h-1}(u_{2,k}(x+h) - u_{2,k}(x))}{h} - \sum_{k=0}^{\infty}(P(T^k(x)))'\beta_{k-1}(x)^2 + 4\sum_{k=0}^{\infty} P(T^k(x))\beta_{k-1}(x)^3 q_{k-1}\Big| \\ \leq \Big|\sum_{k=K_h}^{\infty} (P(T^k(x)))'\beta_{k-1}(x)^2 - 4\sum_{k=K_h}^{\infty} P(T^k(x))\beta_{k-1}(x)^3 q_{k-1}\Big| \\ + |\hat{A}S_1(h)| + |\hat{B}S_2(h)| + |\hat{C}S_3(h)| + |\hat{D}S_4(h)|. \quad (3.1.32)$$

The first term converges to 0 as $h \to 0$ by Lemma 3.16. Then applying Proposition 2.3 (1) and Lemma 3.9, we obtain

$$|S_1(h)| + |S_2(h)| + |S_3(h)| + |S_4(h)| \\ \leq \frac{22}{q_{K_h}}\sum_{k=0}^{K_h-1}\frac{1}{q_k} + \frac{22}{q_{K_h}}\sum_{k=0}^{K_h-1}\frac{1}{q_k} + \frac{22}{q_{K_h}}\sum_{k=0}^{K_h-1}\frac{1}{q_k} + \frac{22}{q_{K_h}}\sum_{k=0}^{K_h-1}\frac{1}{q_k} \xrightarrow[h\to 0]{} 0.$$

It shows that the expression in (3.1.32) converges to 0 as $h \to 0$ which completes the proof of Lemma 3.15. □

Lemma 3.17. *Let $x \in (0,1) \setminus \mathbb{Q}$, then*

$$\frac{\sum_{k=0}^{K_h-1}(u_{3,k}(x+h) - u_{3,k}(x))}{h} \\ \to \sum_{k=0}^{\infty}\Big(\beta_k(x)^2 T^k(x) F_{2,3}(T^{k+1}(x)) + \beta_{k-1}(x)^2\int_0^{p(k)} t^2 F_{2,3}(T(t))dt \\ + 4(-1)^{k+1}\int_0^{T^k(x)} t^2(T^k(x) - 2t)F_{2,3}(T(t))dt \cdot \beta_{k-1}(x)^4 \sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\Big),$$

as $h \to 0$ where $p(k)$ is the smaller endpoint of the interval $I_k(x)$, that is $p(k) = \frac{p_k}{q_k}$ if k is even, and $p(k) = \frac{p_k + p_{k-1}}{q_k + q_{k-1}}$ if k is odd.

Before proving Lemma 3.17 we will prove some claims and lemmas, which then we will use in the proof of Lemma 3.17. First note that for all $k \leq K_h$ the function $u_{3,k}$ is continuous and differentiable on $I_{K_h}(x)$. For brevity, write $\mathcal{I}_k(x) = \int_0^{T^k(x)} t^2(T^k(x) - 2t)F_{2,3}(T(t))dt$. We will now calculate the derivative of $u_{3,k}$. We begin by calculating the derivative of $\mathcal{I}_k(x)$.

Claim 3.18. Let $x \in (0,1) \setminus \mathbb{Q}$. For all $k \in \mathbb{N}$ we have

$$\mathcal{I}_k'(x) = \frac{(-1)^{k+1}}{\beta_{k-1}(x)^2} \int_0^{p(k)} t^2 F_{2,3}(T(t))dt + \frac{(-1)^{k+1}}{\beta_{k-1}(x)^2} T^k(x)^3 F_{2,3}(T^{k+1}(x)),$$

where $p(k) \in \mathbb{Q}$ is the smaller endpoint of the interval $I_k(x)$, that is $p(k) = \frac{p_k}{q_k}$ if k is even, and $p(k) = \frac{p_k + p_{k-1}}{q_k + q_{k-1}}$ if k is odd.

Proof. We use the substitution $y = T^k(x)$, hence $\frac{dy}{dx} = \frac{(-1)^k}{\beta_{k-1}(x)^2}$, and we have

$$\begin{aligned}
\mathcal{I}_k'(x) =& \frac{d}{dx} \int_0^{T^k(x)} t^2(T^k(x) - 2t)F_{2,3}(T(t))dt = \frac{(-1)^k}{\beta_{k-1}(x)^2} \frac{d}{dy} \int_0^y t^2(y - 2t)F_{2,3}(T(t))dt \\
=& \frac{(-1)^k}{\beta_{k-1}(x)^2} \frac{d}{dy} \int_0^{p(k)} t^2(y - 2t)F_{2,3}(T(t))dt + \frac{(-1)^k}{\beta_{k-1}(x)^2} \frac{d}{dy} \int_{p(k)}^y t^2(y - 2t)F_{2,3}(T(t))dt \\
=& \frac{(-1)^k}{\beta_{k-1}(x)^2} \int_0^{p(k)} t^2 F_{2,3}(T(t))dt + \frac{(-1)^k}{\beta_{k-1}(x)^2} y^2(y - 2y)F_{2,3}(T(y)) \\
=& \frac{(-1)^k}{\beta_{k-1}(x)^2} \int_0^{p(k)} t^2 F_{2,3}(T(t))dt + \frac{(-1)^{k+1}}{\beta_{k-1}(x)^2} T^k(x)^3 F_{2,3}(T^{k+1}(x)),
\end{aligned}$$

by the Fundamental Theorem of Calculus and the fact that $t^2(T^k(x) - 2t)F_{2,3}(T(t))dt$ is continuous on $(p(k), T^k(x)]$. □

Claim 3.19. Let $x \in (0,1) \setminus \mathbb{Q}$. For all $k \in \mathbb{N}$ we have

$$\begin{aligned}
u_{3,k}'(x) = \beta_k(x)^2 T^k(x) F_{2,3}(T^{k+1}(x)) + \beta_{k-1}(x)^2 \int_0^{p(k)} t^2 F_{2,3}(T(t))dt \\
+ 4(-1)^{k+1}\mathcal{I}_k(x)\beta_{k-1}(x)^4 \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
u_{3,k}'(x) =& (-1)^{k+1}\mathcal{I}_k(x)(\beta_{k-1}(x)^4)' + (-1)^{k+1}(\mathcal{I}_k(x))'\beta_{k-1}(x)^4 \\
=& 4(-1)^{k+1}\mathcal{I}_k(x)\beta_{k-1}(x)^4 \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2}
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{k+1}\frac{(-1)^{k+1}}{\beta_{k-1}(x)^2}T^k(x)^3F_{2,3}(T^{k+1}(x))\beta_{k-1}(x)^4\\
&+(-1)^{k+1}\frac{(-1)^{k+1}}{\beta_{k-1}(x)^2}\int_0^{p(k)}t^2F_{2,3}(T(t))dt\beta_{k-1}(x)^4 \qquad\qquad \text{by Claim 3.18}\\
=&\beta_k(x)^2T^k(x)F_{2,3}(T^{k+1}(x))+\beta_{k-1}(x)^2\int_0^{p(k)}t^2F_{2,3}(T(t))dt\\
&+4(-1)^{k+1}\mathcal{I}_k(x)\beta_{k-1}(x)^4\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}.
\end{aligned}
$$

This completes the proof of the claim. □

Lemma 3.20. *The series*

$$\sum_{k=0}^{\infty}u'_{3,k}(x)$$

converges for all $x\in(0,1)\setminus\mathbb{Q}$.

Proof. Since $|F_{2,3}|$ is bounded, we have

$$|\mathcal{I}_k(x)|\leq\|F_{2,3}\|_\infty\int_0^{T^k(x)}|t^2(T^k(x)-2t)|dt\leq\|F_{2,3}\|_\infty\int_0^1|t^2||(T^k(x)-2t)|dt\leq\|F_{2,3}\|_\infty, \tag{3.1.33}$$

and

$$
\begin{aligned}
&\Big|\sum_{k=0}^{\infty}\Big(\beta_k(x)^2T^k(x)F_{2,3}(T^{k+1}(x))+\beta_{k-1}(x)^2\int_0^{p(k)}t^2F_{2,3}(T(t))dt\\
&\qquad+4(-1)^{k+1}\mathcal{I}_k(x)\beta_{k-1}(x)^4\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\Big)\Big|\\
\leq&\|F_{2,3}\|_\infty\sum_{k=0}^{\infty}\beta_k(x)^2+\sum_{k=0}^{\infty}\beta_{k-1}(x)^2\int_0^{p(k)}t^2dt\\
&+4\|F_{2,3}\|_\infty\sum_{k=0}^{\infty}\Big|\beta_{k-1}(x)^4\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\Big|\\
\leq&\|F_{2,3}\|_\infty\sum_{k=0}^{\infty}\beta_k(x)^2+\sum_{k=0}^{\infty}\beta_{k-1}(x)^2+4\|F_{2,3}\|_\infty\sum_{k=0}^{\infty}\beta_{k-1}(x)^3q_{k-1} \text{ by Claim 2.6}\\
\leq&(5\|F_{2,3}\|_\infty+1)\sum_{k=0}^{\infty}\frac{1}{q_k^2},
\end{aligned}
$$

by Proposition 2.3 (1). It converges for all $x\in(0,1)\setminus\mathbb{Q}$ by Proposition 2.1 (2). □

We can now prove Lemma 3.17.

Proof of Lemma 3.17. Let $x \in (0,1) \setminus \mathbb{Q}$. By the Mean Value Theorem and the fact that $u_{3,k}$ is continuous and differentiable on $I_k(x)$ for all $k \leq K_h$, we have $\frac{u_{3,k}(x+h)-u_{3,k}(x)}{h} = u'_{3,k}(t_k)$ for some t_k between x and $x+h$ for all $k \leq K_h$. Then $\frac{\sum_{k=0}^{K_h-1}\left(u_{3,k}(x+h)-u_{3,k}(x)\right)}{h} = \sum_{k=0}^{K_h-1} u'_{3,k}(t_k)$. We have

$$\begin{aligned}\Big|\frac{\sum_{k=0}^{K_h-1}(u_{3,k}(x+h)-u_{3,k}(x))}{h} - \sum_{k=0}^{\infty} u'_{3,k}(x)\Big| &= \Big|\sum_{k=0}^{K_h-1} u'_{3,k}(t_k) - \sum_{k=0}^{\infty} u'_{3,k}(x)\Big| \\ &\leq \Big|\sum_{k=0}^{K_h-1}(u'_{3,k}(t_k) - u'_{3,k}(x))\Big| + \Big|\sum_{k=K_h}^{\infty} u'_{3,k}(x)\Big|.\end{aligned}$$

By Lemma 3.20, $|\sum_{k=K_h}^{\infty} u'_{3,k}(x)|$ converges to 0 as $h \to 0$. Then

$$\begin{aligned}&\Big|\sum_{k=0}^{K_h-1}(u'_{3,k}(t_k) - u'_{3,k}(x))\Big| \\ &\qquad\leq \Big|\sum_{k=0}^{K_h-1}\Big(\beta_k(t_k)^2 T^k(t_k) F_{2,3}(T^{k+1}(t_k)) - \beta_k(x)^2 T^k(x) F_{2,3}(T^{k+1}(x))\Big)\Big| \\ &\qquad+ \Big|\sum_{k=0}^{K_h-1}\Big(\beta_{k-1}(t_k)^2\int_0^{p(k,t_k)} t^2 F_{2,3}(T(t))dt - \beta_{k-1}(x)^2\int_0^{p(k,x)} t^2 F_{2,3}(T(t))dt\Big)\Big| \\ &+ \Big|\sum_{k=0}^{K_h-1} 4(-1)^{k+1}\Big(\mathcal{I}_k(t_k)\beta_{k-1}(t_k)^4\sum_{j=0}^{k-1}(-1)^j\frac{T^j(t_k)}{\beta_j(t_k)^2} - \mathcal{I}_k(x)\beta_{k-1}(x)^4\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\Big)\Big|.\end{aligned} \tag{3.1.34}$$

We will now show that each of these terms converges to 0 as $h \to 0$.

We start with the first term. We have

$$\begin{aligned}&\Big|\sum_{k=0}^{K_h-1}(\beta_k(t_k)^2 T^k(t_k) F_{2,3}(T^{k+1}(t_k)) - \beta_k(x)^2 T^k(x) F_{2,3}(T^{k+1}(x)))\Big| \\ &\qquad\leq \sum_{k=0}^{K_h-1}|\beta_k(t_k)^2 T^k(t_k) - \beta_k(x)^2 T^k(x)||F_{2,3}(T^{k+1}(t_k)| \\ &\qquad\qquad+ \sum_{k=0}^{K_h-1}|F_{2,3}(T^{k+1}(t_k)) - F_{2,3}(T^{k+1}(x))|\beta_k(x)^2 T^k(x).\end{aligned}$$

The function $\beta_k(y)^2 T^k(y)$ is continuous and differentiable on $I_{K_h}(x)$ for all $k \leq K_h$, and by Proposition 2.2 (2) we have $(\beta_k(y)^2 T^k(y))' = 2\beta_k(y) T^k(y)(-1)^k q_k + \beta_k(y)^2(-1)^k\frac{1}{\beta_k(y)^2}$.

Thus for all $y \in I_{K_h}(x)$ we have $|(\beta_k(y)^2T^k(y))'| \le 3$. By the fact that $|F_{2,3}|$, the Mean Value Theorem and Lemma 3.9 we have

$$\sum_{k=0}^{K_h-1} |(\beta_k(t_k)^2T^k(t_k) - \beta_k(x)^2T^k(x)||F_{2,3}(T^{k+1}(t_k)| \le \|F_{2,3}\|_\infty \sum_{k=0}^{K_h-1} 3|h| \le 3\|F_{2,3}\|_\infty \frac{1}{q_{K_h}},$$

which converges to 0 as $h \to 0$. Let $N = q_{K_h}^2$. Using the same arguments as in the proof of Lemma 3.11, for some constants c_1, c_2 we have

$$\begin{aligned}
&\sum_{k=0}^{K_h-1} |F_{2,3}(T^{k+1}(t_k)) - F_{2,3}(T^{k+1}(x))|\beta_k(x)^2T^k(x) \\
&\qquad \le c_1 \sum_{k=0}^{K_h-1} |h|\beta_k(x)^2T^k(x)q_{k+1}^2 \log N + c_2 \sum_{k=0}^{K_h-1} \beta_k(x)^2T^k(x)\frac{1}{N^{3/4}} \\
&\qquad \le c_1\frac{2}{q_{K_h}} \sum_{k=0}^{K_h-1} \frac{2\log q_{K_h}}{q_{K_h}} + c_2 \sum_{k=0}^{K_h-1} \frac{1}{q_k^2 q_{K_h}^{3/2}},
\end{aligned}$$

by Proposition 2.3 (1) and Lemma 3.9, which converges to 0 as $h \to 0$.

For the second term, note that since for all $k \le K_h$ we have $t_k \in I_{K_h}(x)$, then for $k \le K_h$ we have that $p(k, t_k) = p(k, x)$. As before, we will denote it $p(k)$. Since $\int_0^{p(k)} t^2F_{2,3}(T(t))dt$ is bounded by $\|F_{2,3}\|_\infty$ for all k, we have

$$\begin{aligned}
&\sum_{k=0}^{K_h-1} \left|\beta_{k-1}(t_k)^2 \int_0^{p(k)} t^2F_{2,3}(T(t))dt - \beta_{k-1}(x)^2 \int_0^{p(k)} t^2F_{2,3}(T(t))dt\right| \\
&\le \|F_{2,3}\|_\infty \sum_{k=0}^{K_h-1} |\beta_{k-1}(t_k)^2 - \beta_{k-1}(x)^2| \le \|F_{2,3}\|_\infty \sum_{k=0}^{K_h-1} 2\frac{q_{k-1}}{q_k}|h| \le 2\|F_{2,3}\|_\infty \frac{1}{q_{K_h}} \sum_{k=0}^{K_h-1} \frac{1}{q_k}.
\end{aligned}$$

The last line follows from the fact that for all $k \le K_h$ the function $\beta_{k-1}(y)^2$ is continuous and differentiable on $I_{K_h}(x)$; by Proposition 2.2 (2) $(\beta_{k-1}(y)^2)' = 2\beta_{k-1}(y)(-1)^{k-1}q_{k-1}$. Then by Proposition 2.3 (1) $|(\beta_{k-1}(y)^2)'| \le 2\frac{q_{k-1}}{q_k}$, for all $y \in I_k(x)$. By the Mean Value Theorem, the fact that $|h| \ge |x - t_k|$ and Lemma 3.9 we obtain the result. It follows from Proposition 2.1 (2) that the term converges to 0 as $h \to 0$.

We consider the last term. Applying Claim 2.6, we get

$$\begin{aligned}
&\left|\sum_{k=0}^{K_h-1} 4(-1)^{k+1}\left(\mathcal{I}_k(t_k)\beta_{k-1}(t_k)^4 \sum_{j=0}^{k-1}(-1)^j\frac{T^j(t_k)}{\beta_j(t_k)^2} - \mathcal{I}_k(x)\beta_{k-1}(x)^4 \sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\right)\right| \\
&\qquad \le 4\sum_{k=0}^{K_h-1} q_{k-1}|\mathcal{I}_k(t_k) - \mathcal{I}_k(x)|\beta_{k-1}(t_k)^3 + 4\sum_{k=0}^{K_h-1} q_{k-1}|\beta_{k-1}(t_k)^3 - \beta_{k-1}(x)^3||\mathcal{I}_k(x)|.
\end{aligned}$$

By Proposition 2.3 (1) and the fact that $|F_{2,3}|$ is bounded, we have

$$4\sum_{k=0}^{K_h-1} q_{k-1}|\mathcal{I}_k(t_k)-\mathcal{I}_k(x)|\beta_{k-1}(t_k)^3 \leq 4\sum_{k=0}^{K_h-1}\frac{q_{k-1}}{q_k^3}|\mathcal{I}_k(t_k)-\mathcal{I}_k(x)|$$

$$\leq 4\sum_{k=0}^{K_h-1}\frac{q_{k-1}}{q_k^3}\Big(\Big|\int_{T^k(x)}^{T^k(t_k)} -2t^3F_{2,3}(T(t))dt\Big| + \Big|\int_0^{T^k(x)} t^2(T^k(t_k)-T^k(x))F_{2,3}(T(t))dt\Big|$$

$$+\Big|\int_{T^k(x)}^{T^k(t_k)} t^2T^k(t_k)F_{2,3}(T(t))dt\Big|\Big)$$

$$\leq 4\|F_{2,3}\|_\infty\sum_{k=0}^{K_h-1}\frac{q_{k-1}}{q_k^3}\Big(2\Big|\int_{T^k(x)}^{T^k(t_k)} t^3dt\Big| + \Big|\int_0^{T^k(x)} t^2(T^k(t_k)-T^k(x))dt\Big| + \Big|\int_{T^k(x)}^{T^k(t_k)} t^2dt\Big|\Big)$$

$$\leq 4\|F_{2,3}\|_\infty\sum_{k=0}^{K_h-1}\frac{q_{k-1}}{q_k^3}\Big(\frac{|T^k(t_k)^4-T^k(x)^4|}{2} + \int_0^{T^k(x)} t^2|T^k(t_k)-T^k(x)|dt$$

$$+\frac{|T^k(t_k)^3-T^k(x)^3|}{3}\Big).$$

By Proposition 2.5, the functions $T^k(y)^4$, $T^k(y)^3$ and $T^k(y)$ are continuous and differentiable on $I_{K_h}(x)$ for all $k \leq K_h$ with $(T^k(y)^4)' = 4(-1)^k\frac{T^k(y)^3}{\beta_{k-1}(y)^2}$, $(T^k(y)^3)' = 3(-1)^k\frac{T^k(y)^2}{\beta_{k-1}(y)^2}$ and $(T^k(y))' = \frac{(-1)^k}{\beta_{k-1}(y)^2}$. It follows that for $y \in I_{K_h}(x)$ we have $|(T^k(y)^4)'| \leq 16q_k^2$, $|(T^k(x)^3)'| = 12q_k^2$ and $|(T^k(x))'| = 4q_k^2$. By the Mean Value Theorem, the fact that $|t_k - x| \leq |h|$ and Lemma 3.9 we get

$$4\sum_{k=0}^{K_h-1} q_{k-1}|\mathcal{I}_k(t_k)-\mathcal{I}_k(x)|\beta_{k-1}(t_k)^3 \leq 4\|F_{2,3}\|_\infty|h|\sum_{k=0}^{K_h-1}\frac{q_{k-1}}{q_k}\Big(8+4\int_0^{T^k(x)} t^2dt+4\Big)$$

$$\leq 48\|F_{2,3}\|_\infty|h|\sum_{k=0}^{K_h-1}\frac{q_{k-1}}{q_k} \leq 48\frac{1}{q_{K_h}}\sum_{k=0}^{K_h-1}\frac{1}{q_k},$$

which converges to 0 as $h \to 0$ by Proposition 2.1 (2). Also, for all $k \leq K_h$ the function $\beta_{k-1}(y)^3$ is continuous and differentiable on $I_{K_h}(x)$ and $(\beta_{k-1}(y)^3)' = 3(-1)^{k-1}\beta_{k-1}(y)^2q_{k-1}$. Hence, $|(\beta_{k-1}(y)^3)'| \leq 3\frac{q_{k-1}}{q_k^2}$ for all $y \in I_k(x)$. By (3.1.33) and Lemma 3.9, we have

$$4\sum_{k=0}^{K_h-1} q_{k-1}|\beta_{k-1}(t_k)^3-\beta_{k-1}(x)^3||\mathcal{I}_k(x)| \leq 12\|F_{2,3}\|_\infty\sum_{k=0}^{K_h-1} q_{k-1}|h|\frac{q_{k-1}}{q_k^2}$$

$$\leq 12\|F_{2,3}\|_\infty\frac{1}{q_{K_h}}\sum_{k=0}^{K_h-1}\frac{1}{q_k},$$

which converges to 0 as $h \to 0$ by Proposition 2.1 (2).

This shows that (3.1.34) converges to 0 as $h \to 0$ completing the proof of Lemma 3.17. $\square$

We are now ready to prove Therem 1.3 (i). For the convenience of the reader, we recall it.

Theorem 1.3. *(i) If $x \in \mathbb{R} \setminus \mathbb{Q}$ is a square-Brjuno number satisfying $(*)$ or $(**)$, then $F_{2,3}$ is differentiable at x. On the other hand, if $x \in \mathbb{R} \setminus \mathbb{Q}$ is not a square-Brjuno number, then $F_{2,3}$ is not differentiable at x.*

Proof. Let $x \in (0,1) \setminus \mathbb{Q}$ be a square-Brjuno number satisfying $(*)$ or $(**)$. By (3.1.23) and Lemmas 3.11, 3.12, 3.15 and 3.17 we conclude that $F_{2,3}$ is differentiable at x and

$$\begin{aligned}
F'_{2,3}(x) =& \lim_{h\to 0} \frac{F_{2,3}(x+h) - F_{2,3}(x)}{h} \\
=& \frac{\pi^3}{3} \sum_{k=0}^{\infty} \beta_{k-1}(x)\gamma_k(x) + \frac{4\pi^3}{3} \sum_{k=0}^{\infty} \left((-1)^k \beta_{k-1}(x)^2 \beta_k(x) \gamma_k(x) \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2} \right) \\
&- \frac{\pi^3}{3} \sum_{k=0}^{\infty} \beta_{k-1}(x)^2 + \sum_{k=0}^{\infty} (P(T^k(x)))' \beta_{k-1}(x)^2 \\
&+ 4 \sum_{k=0}^{\infty} (-1)^k P(T^k(x)) \beta_{k-1}(x)^4 \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2} \\
&+ 6 \sum_{k=0}^{\infty} \Bigg(\beta_k(x)^2 T^k(x) F_{2,3}(T^{k+1}(x)) + \beta_{k-1}(x)^2 \int_0^{p(k)} t^2 F_{2,3}(T(t)) dt \\
&+ 4(-1)^{k+1} \int_0^{T^k(x)} t^2 (T^k(x) - 2t) F_{2,3}(T(t)) dt \cdot \beta_{k-1}(x)^4 \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2} \Bigg),
\end{aligned}$$

where $(P(T^k(x)))'$ is the derivative of the polynomial P evaluated at $T^k(x)$ and $p(k)$ is the smaller endpoint of the interval $I_k(x)$, that is $p(k) = \frac{p_k}{q_k}$ if k is even, and $p(k) = \frac{p_k + p_{k-1}}{q_k + q_{k-1}}$ if k is odd.

We formally showed that if x is square-Brjuno and satisfies $(*)$ or $(**)$, then $F_{2,3}(x) = \frac{\pi^3}{3} \sum_{k=0}^{\infty} u_{1,k}(x) + \sum_{k=0}^{\infty} u_{2,k}(x) + 6 \sum_{k=0}^{\infty} u_{3,k}(x)$ is differentiable at x and

$$F'_{2,3}(x) = \frac{\pi^3}{3} \sum_{k=0}^{\infty} u'_{1,k}(x) + \sum_{k=0}^{\infty} u'_{2,k}(x) + 6 \sum_{k=0}^{\infty} u'_{3,k}(x).$$

Suppose now that $x \in (0,1) \setminus \mathbb{Q}$ is not square-Brjuno. We will show that there exists a sequence $h_n \to 0$ such that $\frac{F_{2,3}(x+h_n) - F_{2,3}(x)}{h_n} \to \infty$ as $n \to \infty$. For each $n \in \mathbb{N}$ odd choose $h_n > 0$ such that if $x \in I(a_1, a_2, ..., a_n, a_{n+1})$, then $x + h_n \in I(a_1, a_2, ..., a_n, a_{n+1} + 2) \setminus \mathbb{Q}$. We have $x + h_n \in I_k(x)$, but $x + h_n \notin I_{n+1}(x)$, and $h_n \to 0$ as $n \to \infty$, and $[x, x + h_n]$

contains the basic interval $I(a_1, a_2, ..., a_n, a_{n+1} + 1)$. We also note that if $t \in [x, x + h_n]$ then

$$q_{n+1} \leq q_{n+1}(t) \leq 3q_{n+1}, \tag{3.1.35}$$

which implies that

$$\frac{1}{18q_{n+1}^2} < \frac{1}{q_{n+1}(t)(q_{n+1}(t)+q_n)} = |I(a_1, a_2, ..., a_n, a_{n+1}+1)| < h_n \leq \frac{1}{q_n q_{n+1}}. \tag{3.1.36}$$

By Equation (3.1.18), we have

$$\begin{aligned}\frac{F_{2,3}(x+h) - F_{2,3}(x)}{h_n} &= \frac{\frac{\pi^3}{3}\sum_{k=0}^{\infty}\left(u_{1,k}(x+h_n) - u_{1,k}(x)\right)}{h_n} \\ &+ \frac{\sum_{k=0}^{\infty}\left(u_{2,k}(x+h_n) - u_{2,k}(x)\right)}{h_n} + \frac{6\sum_{k=0}^{\infty}\left(u_{3,k}(x+h_n) - u_{3,k}(x)\right)}{h_n}.\end{aligned}$$

We will now show that the last two terms converge to some finite limits as $n \to \infty$.

Since $\sum_{k=0}^{\infty} u_{2,k}(y)$ converges absolutely for all y, we have $\sum_{k=0}^{\infty}(u_{2,k}(x+h_n)-u_{2,k}(x)) = \sum_{k=0}^{n}(u_{2,k}(x+h_n) - u_{2,k}(x)) + \sum_{k=n+1}^{\infty}(u_{2,k}(x+h_n) - u_{2,k}(x))$. By the same arguments as in the proof of Lemma 3.15, we conclude that $\frac{\sum_{k=0}^{n} u_{2,k}(x+h_n)-u_{2,k}(x)}{h_n}$ converges to some finite limit as $n \to \infty$. By Proposition 2.3 (1) and since $0 \leq |P(y)| \leq \|P\|_\infty$ for all $y \in (0, 1)$, we have

$$\begin{aligned}&\left|\frac{\sum_{k=n+1}^{\infty}\left(u_{2,n}(x+h_k) - u_{2,k}(x)\right)}{h_n}\right| \\ &\qquad \leq \frac{\sum_{k=n+1}^{\infty}\left(|P(T^k(x+h_n))|\beta_{k-1}(x+h_n)^4 + |P(T^k(x))|\beta_{k-1}(x)^4\right)}{h_n} \\ &\qquad \leq \frac{\|P\|_\infty}{h_n}\sum_{k=n+1}^{\infty}\left(\frac{1}{(q_k(x+h_n))^4} + \frac{1}{q_k^4}\right) \leq 18\|P\|_\infty \sum_{k=n+1}^{\infty}\left(\frac{1}{(q_k(x+h_n))^2} + \frac{1}{q_k^2}\right)\end{aligned}$$

by (3.1.35) and (3.1.36). It converges to 0 as $n \to \infty$ by Proposition 2.1 (2).

Since $\sum_{k=0}^{\infty} u_{3,k}(y)$ converges absolutely for all y, we have $\sum_{k=0}^{\infty}(u_{3,k}(x+h_n)-u_{3,k}(x)) = \sum_{k=0}^{n}(u_{3,k}(x + h_n) - u_{3,k}(x)) + \sum_{k=n+1}^{\infty}(u_{3,k}(x + h_n) - u_{3,k}(x))$. By the same arguments as in Lemma 3.17 we conclude that $\frac{\sum_{k=0}^{n}(u_{3,k}(x+h_n)-u_{3,k}(x))}{h_n}$ converges to some finite limit as $n \to \infty$. By Proposition 2.3 (1) and since $|F_{2,3}|$ is bounded, we have

$$\begin{aligned}\left|\frac{\sum_{k=n+1}^{\infty}(u_{3,k}(x+h_n) - u_{1,k}(x))}{h_n}\right| &\leq \frac{\|F_{2,3}\|_\infty}{h_n}\sum_{k=n+1}^{\infty}\left(\frac{1}{(q_k(x+h_n))^4} + \frac{1}{q_k^4}\right) \\ &\leq 18\|F_{2,3}\|_\infty \sum_{k=n+1}^{\infty}\left(\frac{1}{(q_k(x+h_n))^2} + \frac{1}{q_k^2}\right),\end{aligned}$$

by (3.1.35) and (3.1.36). It converges to 0 as $n \to \infty$ by Proposition 2.1 (2).

Since $\sum_{k=0}^{\infty} u_{1,k}(y)$ converges absolutely for all y, we have $\sum_{k=0}^{\infty}(u_{1,k}(x+h_n)-u_{1,k}(x)) = \sum_{n=0}^{n}(u_{1,k}(x+h_n)-u_{1,k}(x)) + \sum_{k=n+1}^{\infty}(u_{1,k}(x+h_n)-u_{1,k}(x))$. By Proposition 2.3 (1) and (2), we have

$$\begin{aligned} \left|\frac{\sum_{k=n+1}^{\infty}(u_{1,k}(x+h_n)-u_{1,k}(x))}{h_n}\right| &\le \frac{1}{h_n}\sum_{k=n+1}^{\infty}\Big(\frac{\log(2q_{k+1}(x+h_n))}{(q_k(x+h_n))^3 q_{k+1}(x+h_n)} + \frac{\log(2q_{k+1})}{q_k^3 q_{k+1}}\Big) \\ &\le 36\sum_{k=n+1}^{\infty}\Big(\frac{1}{q_k(x+h_n)} + \frac{1}{q_k}\Big), \end{aligned}$$

by (3.1.35) and (3.1.36). It converges to 0 as $n \to \infty$ by Proposition 2.1 (2).

As in the proof of Lemma 3.12, we have

$$\begin{aligned} \frac{\sum_{k=0}^{n}(u_{1,k}(x+h_n)-u_{2,k}(x))}{h_n} &= -\sum_{k=0}^{n}\beta_{k-1}(x)^2\log(T^k(x+h_n)) \\ &\quad + \sum_{k=0}^{n} 4(-1)^{k+1}\beta_{k-1}(x)^3\beta_k(x)\log(T^k(x+h_n))\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2} \\ &\quad + \sum_{k=0}^{n}(-1)^{k+1}\frac{\beta_{k-1}(x)^3\beta_k(x)(\log(T^k(x+h_n))-\log(T^k(x)))}{h_n} \\ &\quad + \sum_{k=0}^{n}(-1)^{k+1}A_k h_n\log(T^k(x+h_n)) + \sum_{k=0}^{n}(-1)^{k+1}B_k h_n^2\log(T^k(x+h_n)) \\ &\quad + \sum_{k=0}^{n}(-1)^{k+1}C_k h_n^3\log(T^k(x+h_n)), \end{aligned}$$

where A_k, B_k, C_k were defined in (3.1.29). By the same arguments as in the proof of Lemma 3.12, we conclude that the last three sums converge to 0 as $n \to \infty$. We also have that the two series $\sum_{k=0}^{n} 4(-1)^{k+1}\ \beta_{k-1}(x)^3\beta_k(x)\log(T^k(x+h_n))\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}$ and $\sum_{k=0}^{n}\frac{(-1)^{k+1}}{h_n}\beta_{k-1}(x)^3\beta_k(x)\ (\log(T^k(x+h_n))-\log(T^k(x)))$ both converge to finite limits as $n \to \infty$. Finally, we have

$$\begin{aligned} -\sum_{k=0}^{n}\beta_{k-1}(x)^2\log(T^k(x+h_n)) &= \sum_{k=0}^{n}\Big(\beta_{k-1}(x)\gamma_k(x) + \beta_{k-1}(x)^2\log\Big(\frac{T^k(x)}{T^k(x+h_n)}\Big)\Big) \\ &= \sum_{k=0}^{n}\beta_{k-1}(x)\gamma_k(x) + \sum_{k=0}^{n}\beta_{k-1}(x)^2\log\Big(\frac{T^k(x)}{T^k(x+h_n)}\Big). \end{aligned}$$

Since $h_n > 0$, we have $x < x+h_n$. If k is odd then $\log(\frac{T^k(x)}{T^k(x+h_n)}) > 0$, and if k is even then $\log(\frac{T^k(x)}{T^k(x+h_n)}) \ge -\log(4)$ by Proposition 2.2. Thus, we have

$$\sum_{k=0}^{n} -\beta_{k-1}(x)^2 \log(T^k(x+h_n)) \geq \sum_{k=0}^{n} \beta_{k-1}(x)\gamma_k(x) - \sum_{\substack{k=0,\\ k \text{ odd}}}^{n} \beta_{k-1}(x)^2 \log(4),$$
$$\geq \sum_{k=0}^{n} \beta_{k-1}(x)\gamma_k(x) - \log(4)\sum_{k=0}^{\infty} \beta_{k-1}(x)^2.$$

By Propositions 2.3 (1) and 2.1 (2) we have $|-\log(4)\sum_{k=0}^{\infty}\beta_{k-1}(x)^2| < \infty$. Since x is not square-Brjuno, $-\sum_{k=0}^{n}\beta_{k-1}(x)^2\log(T^k(x+h_n)) \to \infty$ as $n \to \infty$.

This shows that $\frac{F_{2,3}(x+h_n)-F_{2,3}(x)}{h_n} \to \infty$ as $n \to \infty$, and we conclude that $F_{2,3}$ is not differentiable at x. This completes the proof of Theorem 1.3 (i). □

3.1.5 Proof of Theorem 1.3 (ii)

Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Since $G_{2,3}$ is 1-periodic, we may assume $x \in (0,1)$. For brevity, let

$$\begin{aligned} v_{1,k}(x) &= \beta_{k-1}(x)\beta_k(x)^2\gamma_k(x) \\ v_{2,k}(x) &= Q(T^k(x))\beta_{k-1}(x)^4 \\ v_{3,k}(x) &= \beta_{k-1}(x)^4 \int_0^{T^k(x)} t^2(T^k(x)-2t)G_{2,3}(T(t))dt \end{aligned}$$

By Corollary 3.8, with this notation, for all $n \in \mathbb{N}$, we have

$$G_{2,3}(x) = G_{2,3}(T^n(x))\beta_{n-1}(x)^4 + \pi^2\sum_{k=0}^{n} v_{1,k}(x) + \sum_{k=0}^{n} v_{2,k}(x) + 6\sum_{k=0}^{n} v_{3,k}(x). \tag{3.1.37}$$

For each h, let $K_h \in \mathbb{N}$ such that $x+h \in I_k(x)$ for all $k \leq K_h$ and $x+h \notin I_{K_h+1}(x)$. We then have

$$\begin{aligned} &\frac{G_{2,3}(x+h)-G_{2,3}(x)}{h} \\ &\quad = \frac{\left(G_{2,3}(T^{K_h-1}(x+h))\beta_{K_h-2}(x+h)^4 - G_{2,3}(T^{K_h-1}(x))\beta_{K_h-2}(x)^4\right)}{h} \\ &\quad + \frac{\pi^2\sum_{k=0}^{K_h-1}(v_{1,k}(x+h)-v_{1,k}(x))}{h} + \frac{\sum_{k=0}^{K_h-1}(v_{2,k}(x+h)-v_{2,k}(x))}{h} \\ &\quad + \frac{6\sum_{k=0}^{K_h-1}(v_{3,k}(x+h)-v_{3,k}(x))}{h}. \end{aligned} \tag{3.1.38}$$

We proceed as in the proof of Part (i) of Theorem 1.3. We consider each summand as $h \to 0$.

Lemma 3.21. *Let $x \in (0,1) \setminus \mathbb{Q}$ such that it satisfies (*) or (**), then*

$$\frac{G_{2,3}(T^{K_h-1}(x+h))\beta_{K_h-2}(x+h)^4 - G_{2,3}(T^{K_h-1}(x))\beta_{K_h-2}(x)^4}{h} \to 0,$$

as $h \to 0$.

Proof. The proof is very similar to the proof of the Lemma 3.11, and therefore omitted. The difference is that instead of the sum-to-product identity for the sine function we use the sum-to-product identity for the cosine. □

Lemma 3.22. *Let $x \in (0,1) \setminus \mathbb{Q}$, then*

$$\frac{\sum_{k=0}^{K_h-1}(v_{1,k}(x+h)-v_{1,k}(x))}{h} \to 2\sum_{k=0}^{\infty}(-1)^k\beta_k(x)\gamma_k(x)$$
$$+4\sum_{k=0}^{\infty}\left(\beta_{k-1}(x)\beta_k(x)^2\gamma_k(x)\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\right)+\sum_{k=0}^{\infty}(-1)^{k+1}\beta_{k-1}(x)\beta_k(x)<\infty,$$

as $h \to 0$.

Proof. The proof is very similar to the proof of Lemma 3.12, and therefore omitted. The difference is that we have an additional factor of $T^k(x)$ and no $(-1)^k$ in $v_{1,k}$. □

Lemma 3.23. *Let $x \in (0,1) \setminus \mathbb{Q}$, then*

$$\frac{\sum_{k=0}^{K_h-1}(v_{2,k}(x+h)-v_{2,k}(x))}{h}$$
$$\to \sum_{k=0}^{\infty}(-1)^k(Q(T^k(x)))'\beta_{k-1}(x)^2+4\sum_{k=0}^{\infty}\left(Q(T^k(x))\beta_{k-1}(x)^4\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\right)<\infty,$$

as $h \to 0$, where $(Q(T^k(x)))'$ is the derivative of the polynomial Q evaluated at $T^k(x)$.

Proof. The proof is very similar to the proof of the Lemma 3.15, and therefore omitted. The difference is that we have an additional factor of $(-1)^k$ in $v_{2,k}$. □

Lemma 3.24. *Let $x \in (0,1) \setminus \mathbb{Q}$, then*

$$\frac{\sum_{k=0}^{K_h-1}(v_{3,k}(x+h)-v_{3,k}(x))}{h}$$
$$\to \sum_{k=0}^{\infty}\Bigg((-1)^{k+1}\beta_k(x)^2T^k(x)G_{2,3}(T^{k+1}(x))+(-1)^{k+1}\beta_{k-1}(x)^2\int_0^{p(k)}t^2G_{2,3}(T(t))dt$$
$$+4\int_0^{T^k(x)}t^2(T^k(x)-2t)G_{2,3}(T(t))dt\cdot\beta_{k-1}(x)^4\sum_{j=0}^{k-1}(-1)^j\frac{T^j(x)}{\beta_j(x)^2}\Bigg)<\infty,$$

as $h \to 0$ where $p(k)$ is the smaller endpoint of the interval $I_k(x)$, that is $p(k)=\frac{p_k}{q_k}$ if k is even, and $p(k)=\frac{p_k+p_{k-1}}{q_k+q_{k-1}}$ if k is odd.

Proof. The proof is very similar to the proof of the Lemma 3.17, and therefore omitted. The difference is that we do not have the factor $(-1)^k$ in $v_{3,k}$. □

We are now ready to prove Therem 1.3 (i). For the convenience of the reader, we recall it.

Theorem 1.3. *(ii) If $x \in \mathbb{R} \setminus \mathbb{Q}$ satisfies $(*)$ or $(**)$, then $G_{2,3}$ is differentiable at x.*

Proof. Let $x \in (0,1)\setminus\mathbb{Q}$ satisfy $(*)$ or $(**)$. By (3.1.38) and Lemmas 3.21-3.24 we conclude that $G_{2,3}$ is differentiable at x and

$$\begin{aligned}
G'_{2,3}(x) = &\lim_{h\to 0} \frac{G_{2,3}(x+h) - G_{2,3}(x)}{h} \\
=&2\pi^2 \sum_{k=0}^{\infty} (-1)^k \beta_k(x)\gamma_k(x) + 4\pi^2 \sum_{k=0}^{\infty} \left(\beta_{k-1}(x)\beta_k(x)^2\gamma_k(x) \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2} \right) \\
&+ \pi^2 \sum_{k=0}^{\infty} (-1)^{k+1} \beta_{k-1}(x)\beta_k(x) + \sum_{k=0}^{\infty} (-1)^k (Q(T^k(x)))' \beta_{k-1}(x)^2 \\
&+ 4 \sum_{k=0}^{\infty} \left(Q(T^k(x))\beta_{k-1}(x)^4 \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2} \right) \\
&+ 6 \sum_{k=0}^{\infty} \Bigg((-1)^{k+1} \beta_k(x)^2 T^k(x) G_{2,3}(T^{k+1}(x)) + (-1)^{k+1} \beta_{k-1}(x)^2 \int_0^{p(k)} t^2 G_{2,3}(T(t))dt \\
&+ 4 \int_0^{T^k(x)} t^2 (T^k(x) - 2t) G_{2,3}(T(t))dt \cdot \beta_{k-1}(x)^4 \sum_{j=0}^{k-1} (-1)^j \frac{T^j(x)}{\beta_j(x)^2} \Bigg)
\end{aligned}$$

where $(Q(T^k(x)))'$ is the derivative of the polynomial Q evaluated at $T^k(x)$ and $p(k)$ is the smaller endpoint of the interval $I_k(x)$, that is $p(k) = \frac{p_k}{q_k}$ if k is even, and $p(k) = \frac{p_k + p_{k-1}}{q_k + q_{k-1}}$ if k is odd. $\square$

3.1.6 Functional equation for φ_k

In order to prove Conjecture 1.5 for $k \geq 4$, we would proceed as in the case $k = 2$. There are a lot of terms to analyse, but we believe that for any given $k \geq 4$ this method would work (adding a technical condition similar to $(*)$ of the type $\frac{\log(q_{n+4})}{q_n^k} \to 0$). However the calculations become very long, and we do not do it explicitly. We present arguments justifying the conjecture. We start by finding the functional equation for φ_k.

Theorem 3.25. *For $k \geq 4$ even, for $\alpha \in \mathbb{H}$, and $\tau \in \mathbb{H}$, we have*

$$\varphi_k(\tau) = \tau^{k+2} \varphi_k\left(-\frac{1}{\tau}\right) - \frac{k}{C_k} \tau \mathrm{Log}(\tau) + P_{k,\alpha}(\tau) + \int_\alpha^\tau Q_{k,\alpha}(t,\tau)\varphi_k\left(-\frac{1}{t}\right) dt, \tag{3.1.39}$$

where Log *denotes the principal value of the complex logarithm, $P_{k,\alpha}(\tau)$ is a polynomial in τ of degree less than or equal to $k+1$ depending on α, $Q_{k,\alpha}(t,\tau)$ is a polynomial in t and τ of degree less than or equal to $k+1$ also depending on α, and $C_k = -\frac{k!2k}{(2i\pi)^{k+1}B_k}$.*

We divide the proof into various lemmas. Let $4 \leq k \in \mathbb{N}$ even, $\alpha \in \mathbb{H}$ and $\tau \in \mathbb{H}$ be all fixed. We make the following observations.

Claim 3.26. We have

$$C_k \cdot \varphi_k(\tau) = \int_{i\infty}^{\tau} (\tau - t)^k (E_k(t) - 1) dt, \tag{3.1.40}$$

where $C_k = -\frac{k!2k}{(2i\pi)^{k+1} B_k}$.

Proof. It follows by integrating the right-hand side of Equation (3.1.40) by parts k times. $\square$

Claim 3.27. For $1 \leq j \leq k+1$ we have

$$\varphi_k^{(j)}(\tau) = (2\pi i)^j \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^{k+1-j}} e^{2i\pi n\tau}, \tag{3.1.41}$$

in particular

$$\varphi_k^{(k+1)}(\tau) = \frac{k!}{C_k}(E_k(\tau) - 1), \tag{3.1.42}$$

where C_k is as in the Claim 3.26.

Proof. We obtain (3.1.41) by differentiating $\varphi_k(\tau)$ j times. Equality (3.1.42) follows from (3.1.41) and the definition of Eisenstein series. $\square$

Claim 3.28. We have

$$C_k \cdot \varphi_k(\tau) = \int_{\alpha}^{\tau} (\tau - t)^k E_k(t) dt + p_{k,\alpha}(\tau),$$

where $p_{k,\alpha}(\tau)$ is a polynomial in τ of degree less than or equal to $k+1$, which depends on α. In particular, $p_{k,\alpha}(\tau) = \frac{(\tau-\alpha)^{k+1}}{k+1} - \sum_{m=0}^{k} \frac{k!(\tau-\alpha)^{k-m}}{(k-m)!(2i\pi)^{k+1}} \varphi_k^{(k-m)}(\alpha)$.

Proof. We note that Claim 3.26 implies that

$$C_k \varphi_k(\tau) = \int_{\alpha}^{\tau} (\tau - t)^k E_k(t) dt - \int_{\alpha}^{\tau} (\tau - t)^k dt + \int_{i\infty}^{\alpha} (\tau - t)^k (E_k(t) - 1) dt. \tag{3.1.43}$$

We have

$$-\int_{\alpha}^{\tau} (\tau - t)^k dt = \frac{(\tau - \alpha)^{k+1}}{k+1}. \tag{3.1.44}$$

Then integrating by parts the last term in (3.1.43) k times gives

$$\int_{i\infty}^{\alpha} (\tau - t)^k (E_k(t) - 1) dt = -\sum_{m=0}^{k} \frac{k!(\tau - \alpha)^{k-m}}{(k-m)!(2i\pi)^{k+1}} \varphi_k^{(k-m)}(\alpha).$$

$\square$

Then by (1.0.1) with $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we get

$$\int_\alpha^\tau (\tau - t)^k E_k(t)dt = \int_\alpha^\tau \frac{(\tau - t)^k}{t^k} E_k\left(-\frac{1}{t}\right) dt.$$

Substituting (3.1.42) with $\tau = -\frac{1}{t}$, we obtain

$$\begin{aligned}\int_\alpha^\tau \frac{(\tau - t)^k}{t^k} E_k\left(-\frac{1}{t}\right) dt &= \int_\alpha^\tau \frac{(\tau - t)^k}{t^k} \left(1 + \frac{C_k}{k!}\varphi_k^{(k+1)}\left(-\frac{1}{t}\right)\right) dt \\ &= \int_\alpha^\tau \frac{(\tau - t)^k}{t^k} dt + \frac{C_k}{k!}\int_\alpha^\tau \frac{(\tau - t)^k}{t^k} \varphi_k^{(k+1)}\left(-\frac{1}{t}\right) dt. \quad (3.1.45)\end{aligned}$$

Claim 3.29. We have

$$\int_\alpha^\tau \frac{(\tau - t)^k}{t^k} dt = -k\tau \mathrm{Log}(\tau) + q_{k,\alpha}(\tau), \tag{3.1.46}$$

where $q_{k,\alpha}(\tau)$ is a polynomial in τ of degree less than or equal to k depending of α. In particular, $q_{k,\alpha}(\tau) = \sum_{m=0}^{k-2} (-1)^m \binom{k}{m} \frac{1}{m-k+1}(\tau - \tau^{k-m}\alpha^{m-k+1}) + k\tau\mathrm{Log}(\alpha) + \tau - \alpha$.

Proof. To see that, we note

$$\int_\alpha^\tau \frac{(\tau - t)^k}{t^k} dt = \int_\alpha^\tau \left(\sum_{m=0}^{k-2} (-1)^m \binom{k}{m} \tau^{k-m} t^{m-k} - k\tau t^{-1} + 1\right) dt.$$

□

Substituting (3.1.44), (3.1.45) and (3.1.46) into (3.1.43) gives

$$\varphi_k(\tau) = -\frac{k}{C_k}\tau\mathrm{Log}(\tau) + \frac{1}{k!}\int_\alpha^\tau \frac{(\tau - t)^k}{t^k} \varphi_k^{(k+1)}\left(-\frac{1}{t}\right) dt + \frac{1}{C_k}(p_{k,\alpha}(\tau) + q_{k,\alpha}(\tau)). \tag{3.1.47}$$

It rests to evaluate the integral $\int_\alpha^\tau \frac{(\tau-t)^k}{t^k} \varphi_k^{(k+1)}\left(-\frac{1}{t}\right) dt$.

Then we have

Claim 3.30. We have

$$\int_\alpha^\tau \frac{(\tau - t)^k}{t^k} \varphi_k^{(k+1)}\left(-\frac{1}{t}\right) dt = k!\tau^{k+2}\varphi_k\left(-\frac{1}{\tau}\right) + r_{k,\alpha}(\tau) + \int_\alpha^\tau s_{k,\alpha}(t,\tau)\varphi_k\left(-\frac{1}{t}\right) dt,$$

where $r_{k,\alpha}(\tau)$ is a polynomial in τ of degree less than or equal to $k+1$ depending on α, and $s_{k,\alpha}(t,\tau)$ is a polynomial in t and τ of degree less than or equal to $k+1$.

Proof. We use the substitution $u = -\frac{1}{t}$, and we have

$$\int_{\alpha}^{\tau} \frac{(\tau - t)^k}{t^k} \varphi_k^{(k+1)} \left(-\frac{1}{t}\right) dt = \int_{-\frac{1}{\alpha}}^{-\frac{1}{\tau}} u^{k-2} \left(\tau + \frac{1}{u}\right)^k \varphi_k^{(k+1)}(u) du.$$

For simplicity, we will define $v_0(x,\tau) = \left(-\frac{1}{x}\right)^{k-2} (\tau - x)^k$, and for $m \geq 0$, $v_m(x,\tau) = \left[\frac{\partial^m u^{k-2}\left(\tau + \frac{1}{u}\right)^k}{\partial u^m}\right]_{u=-\frac{1}{x}}$. By Leibniz product formula for $m \leq k$, we have

$$\begin{aligned} v_m\left(-\frac{1}{u}, \tau\right) &= \frac{\partial^m u^{k-2}\left(\tau + \frac{1}{u}\right)^k}{\partial u^m} \\ &= \sum_{j=0}^{m} (-1)^j \frac{m!k!(j+1)}{(m-j)!(k-m+j)!} u^{-j-2} \tau^{m-j} (\tau u + 1)^{k-m+j} \\ &\quad = \sum_{j=0}^{m} (-1)^j \frac{m!k!(j+1)}{(m-j)!(k-m+j)!} u^{-2+k-m} \tau^{m-j} \left(\tau + \frac{1}{u}\right)^{k-m+j}. \end{aligned}$$

Hence for all $0 \leq m \leq k$, we have $v_m(\tau,\tau) = 0$. Then integrating by parts k times gives

$$\begin{aligned} \int_{-\frac{1}{\alpha}}^{-\frac{1}{\tau}} u^{k-2} \left(\tau + \frac{1}{u}\right)^k \varphi_k^{(k+1)}(u) du = \sum_{i=0}^{k-1} (-1)^{i+1} v_i(\alpha,\tau) \varphi_k^{(k-i)} \left(-\frac{1}{\alpha}\right) \\ + \int_{-\frac{1}{\alpha}}^{-\frac{1}{\tau}} v_k \left(-\frac{1}{u}, \tau\right) \varphi_k'(u) du. \end{aligned}$$

We observe that $\sum_{i=0}^{k-1} (-1)^{i+1} v_i(\alpha,\tau) \varphi_k^{(k-i)} \left(-\frac{1}{\alpha}\right)$ is a polynomial in τ of degree less than or equal to k. We then note that

$$v_k\left(-\frac{1}{u}, \tau\right) = k! u^{-2} \tau^k + \sum_{j=1}^{k} (-1)^j \frac{k!k!(j+1)}{(k-j)!(j)!} u^{-2} \tau^{k-j} \left(\tau + \frac{1}{u}\right)^j.$$

For simplicity, write $w_k\left(-\frac{1}{u}, \tau\right) = \sum_{j=1}^{k} (-1)^j \frac{k!k!(j+1)}{(k-j)!(j)!} u^{-2} \tau^{k-j} \left(\tau + \frac{1}{u}\right)^j$. We then have:

1. $w_k(\tau,\tau) = 0$;
2. $w_k(\alpha,\tau)$ is a polynomial in τ of degree less then or equal to k;
3. $\frac{\partial w_k\left(-\frac{1}{u},\tau\right)}{\partial u} = -\sum_{j=1}^{k} (-1)^j \frac{k!k!(j+1)}{(k-j)!(j)!} \tau^{k-j} u^{-3} \left(2\left(\tau + \frac{1}{u}\right)^j + j u^{-1} \left(\tau + \frac{1}{u}\right)^{j-1}\right)$;
4. $\left[\frac{\partial w_k\left(-\frac{1}{u},\tau\right)}{\partial u}\right]_{u=-\frac{1}{t}}$ can be written as $t^2 w_{k+1,\alpha}(t,\tau)$, where $w_{k+1,\alpha}(t,\tau)$ is a polynomial in t and τ of degree less than or equal to $k+1$.

Therefore we have

$$\int_{-\frac{1}{\alpha}}^{-\frac{1}{\tau}} v_k\left(-\frac{1}{u},\tau\right)\varphi_k'(u)du = k!\tau^{k+2}\varphi_k\left(-\frac{1}{\tau}\right) - k!\alpha^k\varphi_k\left(-\frac{1}{\alpha}\right) - w_k(\alpha,\tau)\varphi_k\left(-\frac{1}{\alpha}\right) + \int_\alpha^\tau \left(s_{k,\alpha}(t,\tau)+2k!t\tau^k\right)\varphi_k\left(-\frac{1}{t}\right)dt.$$

Letting $r_{k,\alpha}(\tau)=\sum_{i=0}^{k-1}(-1)^{i+1}v_i(\alpha,\tau)\varphi_k^{(k-i)}\left(-\frac{1}{\alpha}\right) - k!\alpha^k\varphi_k\left(-\frac{1}{\alpha}\right) - w_k(\alpha,\tau)\varphi_k\left(-\frac{1}{\alpha}\right)$, and $s_{k,\alpha}(t,\tau)=w_{k+1,\alpha}(t,\tau)+2k!t\tau^k$, gives the result. □

Proof of Theorem 3.25. If follows from Equations (3.1.46) and (3.1.47) that for $\alpha\in\mathbb{H}$, and $\tau\in\mathbb{H}$, we have

$$\varphi_k(\tau)=\tau^{k+2}\varphi_k\left(-\frac{1}{\tau}\right)-\frac{k}{C_k}\tau\mathrm{Log}(\tau)+P_{k,\alpha}(\tau)+\int_\alpha^\tau Q_{k,\alpha}(t,\tau)\varphi_k\left(-\frac{1}{t}\right)dt,$$

where $P_{k,\alpha}(\tau)=\frac{1}{C_k}(p_{k,\alpha}(\tau)+q_{k,\alpha}(\tau))+\frac{1}{k!}r_{k,\alpha}(\tau)$ is a polynomial in τ of degree less than or equal to $k+1$, and $Q_{k,\alpha}(t,\tau)=\frac{1}{k!}s_{k,\alpha}(t,\tau)$ is a polynomial in t and τ of degree less than or equal to $k+1$. This completes the proof of Theorem 3.25 □

3.1.7 Heuristic approach to Conjecture 1.5

We assume we can let $\alpha\to 0$. For $x\in\mathbb{R}^+$, letting $\tau\to x$, we get:

$$\varphi_k(x)=x^{k+2}\varphi_k\left(-\frac{1}{x}\right)-\frac{k}{C_k}x\log(x)+P_{k,0}(x)+\int_0^x Q_{k,0}(t,x)\varphi_k\left(-\frac{1}{t}\right)dt. \quad (3.1.48)$$

We read the behaviour of $F_{k,k+1}$ and $G_{k,k+1}$ around 0 from this equation. In order to prove part (i) of Conjecture 1.5, we would find another functional equation for φ_k in a similar way to the proof of Theorem 3.25. We would apply the modular property of E_k with $t\in\mathbb{H}$ and $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$.

Taking imaginary parts on both sides of Equation (3.1.48), we get

$$F_{k,k+1}(x)=x^{k+2}F_{k,k+1}\left(-\frac{1}{x}\right)+D_kx\log(x)+P_k(x)+\int_0^x Q_k(t,x)F_{k,k+1}\left(-\frac{1}{t}\right)dt, \quad (3.1.49)$$

where $Q_k(t,x)=\mathrm{Im}(Q_{k,0}(t,x))$, $P_k(x)=\mathrm{Im}(P_{k,0}(x))$, and $D_k=\frac{(2i)^k\pi^{k+1}B_k}{k!}$. Taking real parts on both sides of Equation (3.1.48), we get

$$G_{k,k+1}(x)=x^{k+2}G_{k,k+1}\left(-\frac{1}{x}\right)+R_k(x)+\int_0^x S_k(t,x)G_{k,k+1}\left(-\frac{1}{t}\right)dt, \quad (3.1.50)$$

where $S_k(t,x)=\mathrm{Re}(Q_{k,0}(t,x))$ and $R_{k+1}(x)=\mathrm{Re}(P_{k,0}(x))$.

Claim 3.31. Let $x \in (0,1)\backslash\mathbb{Q}$. Assume that (3.1.48) holds.

1. We have:

$$F_{k,k+1}(x) = -x^{k+2}F_{k,k+1}(T(x)) + D_k x\log(x) + P_k(x) - \int_0^x Q_k(t,x)F_{k,k+1}(T(t))dt;$$
$$G_{k,k+1}(x) = x^{k+2}G_{k,k+1}(T(x)) + R_k(x) + \int_0^x S_k(t,x)G_{k,k+1}(T(t))dt. \tag{3.1.51}$$

2. For all $n \in \mathbb{N}$, we have

$$\begin{aligned}F_{k,k+1}(x) =&(-1)^{n+1}\beta_n(x)^{k+2}F_{k,k+1}(T^{n+1}(x)) - D_k\sum_{j=0}^{n}(-1)^j\beta_{j-1}(x)^k\beta_j(x)\gamma_j(x)\\&+\sum_{j=0}^{n}(-1)^j\beta_{j-1}(x)^{k+2}P_k(T^j(x))\\&+\sum_{j=0}^{n}(-1)^{j+1}\beta_{j-1}(x)^{k+2}\int_0^{T^j(x)}Q_k(t,T^j(x))F_{k,k+1}(T(t))dt;\end{aligned} \tag{3.1.52}$$
$$\begin{aligned}G_{k,k+1}(x) =&\beta_n(x)^{k+2}G_{k,k+1}(T^{n+1}(x)) + \sum_{j=0}^{n}\beta_{j-1}(x)^{k+2}R_k(T^j(x))\\&+\sum_{j=0}^{n}\beta_{j-1}(x)^{k+2}\int_0^{T^j(x)}S_k(t,T^j(x))G_{k,k+1}(T(t))dt.\end{aligned}$$

3. Letting $n \to \infty$, we have

$$\begin{aligned}F_{k,k+1}(x) =&- D_k\sum_{j=0}^{\infty}(-1)^j\beta_{j-1}(x)^k\beta_j(x)\gamma_j(x)\\&+\sum_{j=0}^{\infty}(-1)^j\beta_{j-1}(x)^{k+2}P_k(T^j(x))\\&+\sum_{j=0}^{\infty}(-1)^{j+1}\beta_{j-1}(x)^{k+2}\int_0^{T^j(x)}Q_k(t,T^j(x))F_{k,k+1}(T(t))dt;\end{aligned} \tag{3.1.53}$$
$$\begin{aligned}G_{k,k+1}(x) =&\sum_{j=0}^{\infty}\beta_{j-1}(x)^{k+2}R_k(T^j(x))\\&+\sum_{j=0}^{\infty}\beta_{j-1}(x)^{k+2}\int_0^{T^j(x)}S_k(t,T^j(x))G_{k,k+1}(T(t))dt.\end{aligned}$$

Proof. Equations 3.1.51 follow from (3.1.49), (3.1.50) and the fact that $F_{k,k+1}\left(-\frac{1}{x}\right) = -F_{k,k+1}(T(x))$ and $G_{k,k+1}\left(-\frac{1}{x}\right) = G_{k,k+1}(T(x))$ for all $x \in (0,1)\setminus\mathbb{Q}$. Iterating the equations in (3.1.51) gives (3.1.52). Then we observe that $(-1)^{n+1}\beta_n(x)^{k+2}F_{k,k+1}(T^{n+1}(x)) \to 0$

and $\beta_n(x)^{k+2}G_{k,k+1}(T^{n+1}(x)) \to 0$ as $n \to \infty$. As the series

$$\sum_{j=0}^{\infty} \Big(D_k(-1)^{j+1}\beta_{j-1}(x)^k\beta_j(x)\gamma_j(x) + (-1)^j\beta_{j-1}(x)^{k+2}P_k(T^j(x)) + (-1)^{j+1}\beta_{j-1}(x)^{k+2}\int_0^{T^j(x)} Q_k(t,T^j(x))F_{k,k+1}(T(t))dt\Big),$$

$$\sum_{j=0}^{\infty} \Big(\beta_{j-1}(x)^{k+2}R_k(T^j(x)) + \beta_{j-1}(x)^{k+2}\int_0^{T^j(x)} S_k(t,T^j(x))G_{k,k+1}(T(t))dt\Big)$$

converge absolutely, we obtain (3.1.53). □

We then have

Claim 3.32. Let $x \in (0,1) \setminus \mathbb{Q}$. For all $j \in \mathbb{N}$ we have that $\beta_{j-1}(x)^k\beta_j(x)\gamma_j(x)$ is differentiable at x and

$$(\beta_{j-1}(x)^k\beta_j(x)\gamma_j(x))' = (-1)^j\beta_{j-1}(x)^{k+2} + (-1)^j(k+2)\beta_{j-1}(x)^{k-1}\beta_j(x)\gamma_j(x)q_{j-1} - (-1)^j\beta_{j-1}(x)^{k-1}\gamma_j(x).$$

We also have

$$\left|\sum_{j=0}^{\infty} \big(\beta_{j-1}(x)^{k+2} + (k+2)\beta_{j-1}(x)^{k-1}\beta_j(x)\gamma_j(x)q_{j-1} - \beta_{j-1}(x)^{k-1}\gamma_j(x)\big)\right| < \infty,$$

if and only if

$$\sum_{j=0}^{\infty} \frac{\log(q_{j+1})}{q_j^k} < \infty.$$

Proof. The first part of the claim follows from Proposition 2.5. We obtain the second part by Proposition 2.3. □

Claim 3.33. Let $x \in (0,1) \setminus \mathbb{Q}$. For all $j \in \mathbb{N}$ we have that $\beta_{j-1}(x)^{k+2}P_k(T^j(x))$ is differentiable at x and

$$(\beta_{j-1}(x)^{k+2}P_k(T^j(x)))' = (-1)^{j-1}(k+2)\beta_{j-1}(x)^{k+1}q_{k-1}P_k(T^j(x)) + (-1)^j\beta_{j-1}(x)^kP_k'(T^j(x)),$$

where $P_k'(T^j(x))$ is the derivative of $P_k(y)$ with respect to y evaluated at $T^j(x)$.

We also have that

$$\left|\sum_{j=0}^{\infty} \big(-(k+2)\beta_{j-1}(x)^{k+1}q_{j-1}P_k(T^j(x)) + \beta_{j-1}(x)^kP_k'(T^j(x))\big)\right| < \infty,$$

for all $x \in (0,1)\setminus\mathbb{Q}$.

Proof. The first part of the claim follows from Proposition 2.5 and the fact that P_k is differentiable on $\mathbb{R}$. We obtain the second part by Proposition 2.3 and the fact that $|P_k|$ and $|P_k'|$ are bounded on $(0, 1)$. □

Claim 3.34. For all $j \in \mathbb{N}$ we have that $\beta_{j-1}(x)^{k+2} \int_0^{T^j(x)} Q_k(t, T^j(x))F_{k,k+1}(T(t))dt$ is differentiable at $x \in (0, 1) \setminus \mathbb{Q}$ and

$$\begin{aligned}
&\Big(\beta_{j-1}(x)^{k+2} \int_0^{T^j(x)} Q_k(t, T^j(x))F_{k,k+1}(T(t))dt\Big)' \\
&\qquad = (-1)^{j-1}(k + 2)\beta_{j-1}(x)^{k+1}q_{j-1} \int_0^{T^j(x)} Q_k(t, T^j(x))F_{k,k+1}(T(t))dt \\
&\qquad\qquad + (-1)^j\beta_{j-1}(x)^k \int_0^{p(j)} Q_k'(t, T^j(x))F_{k,k+1}(T(t))dt \\
&\qquad\qquad\qquad + (-1)^j\beta_{j-1}(x)^k Q_k(t, T^j(x))F_{k,k+1}(T^{j+1}(t)),
\end{aligned}$$

where $Q_k'(t, T^j(x))$ is the derivative of $Q_k(t, y)$ with respect to y evaluated at $y = T^j(x)$, and $p(j)$ is the smaller endpoint of the interval $I_j(x)$.

We also have

$$\begin{aligned}
\Bigg|\sum_{j=0}^{\infty}\Bigg(&(k + 2)\beta_{j-1}(x)^{k+1}q_{j-1} \int_0^{T^j(x)} Q_k(t, T^j(x))F_{k,k+1}(T(t))dt \\
&- \beta_{j-1}(x)^k \int_0^{p(j)} Q_k'(t, T^j(x))F_{k,k+1}(T(t))dt \\
&- \beta_{j-1}(x)^k Q_k(t, T^j(x))F_{k,k+1}(T^{j+1}(t))\Bigg)\Bigg| < \infty,
\end{aligned}$$

for all $x \in (0, 1)\setminus\mathbb{Q}$.

Proof. The first part of the claim follows from Proposition 2.5, the Foundamental Theorem of Calculus and the fact that $Q_k(t, T^j(x))F_{k,k+1}(T(t))dt$ is continuous on $(p(j), T^j(x)]$. We obtain the second part by Proposition 2.3 and the fact that $|F_{k,k+1}|$, $|Q_k|$ and $|Q_k'|$ are bounded on $(0, 1)$. □

Supposing that we can let $\alpha \to 0$ in (3.1.39). The individual terms in the two sums in (3.1.53) are differentiable at every $x \in (0, 1) \setminus \mathbb{Q}$ and the sums of the derivatives evaluated at $x \in (0, 1) \setminus \mathbb{Q}$ converge. Since we are dealing with infinite sums, we cannot say that the derivative of $F_{k,k+1}(x)$ is the sum of derivatives from Claims 3.32-3.34 over $j \in \mathbb{N}$. Formally, to prove Conjecture 1.5 (ii) and (iii), we would proceed as in Sections 3.1.4 and 3.1.5 first showing that we can let $\alpha \to 0$ in (3.1.39).

3.2 Modulus of continuity of $F_{2,3}$

3.2.1 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. For the convenience of the reader, we recall it.

Theorem 1.4. *For all $x \in (0,1) \setminus \mathbb{Q}$ and all $y \in (0,1)$, we have*

$$|F_{2,3}(x) - F_{2,3}(y)| \leq C_1|x-y| \log\left(\frac{1}{|x-y|}\right) + C_2|x-y|, \tag{1.2.1}$$

and

$$|G_{2,3}(x) - G_{2,3}(y)| \leq C_3|x-y| \log\left(\frac{1}{|x-y|}\right) + C_4|x-y|, \tag{1.2.2}$$

for some constants C_1, C_2, C_3, C_4 dependent only on x.

Proof. Let $x \in (0,1)\setminus\mathbb{Q}$, let $y \in (0,1)$ and $K_y \in \mathbb{N}$ such that $y \in I_{K_y}(x)$, and $y \notin I_{K_y+1}(x)$. By Corollary 3.8, we have

$$\begin{aligned}|F_{2,3}(x) - F_{2,3}(y)| &\leq |F_{2,3}(T^{K_y-1}(x))\beta_{K_y-2}(x)^4 - F_{2,3}(T^{K_y-1}(y))\beta_{K_y-2}(y)^4| \\ &+ \frac{\pi^3}{3}\sum_{k=0}^{K_y-1}|u_{1,k}(x) - u_{1,k}(y)| + \sum_{k=0}^{K_y-1}|u_{2,k}(x) - u_{2,k}(y)| + 6\sum_{k=0}^{K_y-1}|u_{3,k}(x) - u_{3,k}(y)|,\end{aligned}$$

where $u_{1,k}, u_{2,k}, u_{3,k}$ were defined in (3.1.22). We will consider each term separately.

Let $N = \lceil \frac{1}{|x-y|^2} \rceil$. By the same arguments as in Lemma 3.11, we have

$$\begin{aligned}&|F_{2,3}(T^{K_y-1}(x))\beta_{K_y-2}(x)^4 - F_{2,3}(T^{K_y-1}(y))\beta_{K_y-2}(y)^4| \\ &\quad\leq |(F_{2,3}(T^{K_y-1}(x)) - F_{2,3}(T^{K_y-1}(y)))\beta_{K_y-2}(y)^4| \\ &\quad\quad + |(\beta_{K_y-2}(x)^4 - \beta_{K_y-2}(y)^4)F_{2,3}(T^{K_y-1}(x))| \\ &\quad\leq \beta_{K_y-2}(y)^4 \sum_{n=1}^{N} \frac{\sigma_1(n)}{n^3}|\sin(2\pi nx) - \sin(2\pi ny)| \\ &\quad + \beta_{K_y-2}(y)^4 \sum_{n=N+1}^{\infty} \frac{\sigma_1(n)}{n^3}|\sin(2\pi nx) - \sin(2\pi ny)| + 4\|F_{2,3}\|_\infty|x-y| \\ &\quad\leq c_1|x-y|q_{K_y-1}^{-2}\log N + c_2\frac{1}{N^{3/4}}\beta_{K_y-2}(y)^4 + 4\|F_{2,3}\|_\infty|x-y| \\ &\quad\leq 2c_1|x-y|\log\left(\frac{1}{|x-y|}\right) + (4\|F_{2,3}\|_\infty + c_2)|x-y|,\end{aligned} \tag{3.2.1}$$

for some constants c_1, c_2 independent of x and y.

We observe that $u_{1,k}, u_{2,k}, u_{3,k}$ are continuous and differentiable on $I_k(x)$ for all $k \leq K_y$. Therefore, by the Mean Value Theorem, for each k there exists t_k between x and y such that

$$\begin{aligned}
&\sum_{k=0}^{K_y-1} |u_{1,k}(x) - u_{1,k}(y)| \\
&\quad = \sum_{k=0}^{K_y-1} |x-y||(4\beta_{k-1}(t_k)^2\beta_k(t_k)q_{k-1} - \beta_{k-1}(t_k)^2)\log(T^k(t_k)) - \beta_{k-1}(t_k)^2| \\
&\quad \leq |x-y|\Big(4(\log(2)+1)\sum_{k=0}^{\infty}\frac{1}{\mathrm{F}_{k+1}} + \sum_{k=0}^{K_y-1}|\beta_{k-1}(t_k)^2\log(T^k(t_k))| + \sum_{k=0}^{\infty}\frac{1}{\mathrm{F}_{k+1}^2}\Big).
\end{aligned}$$

Observe that for all $k \leq K_y - 1$, we have $\frac{q_k}{2q_{k+1}} \leq T^k(t_k) \leq \frac{2q_k}{q_{k+1}}$, and hence $\frac{1}{4}T^k(x) \leq T^k(t_k) \leq 4T^k(x)$. We then have

$$\begin{aligned}
&\sum_{k=0}^{K_y-1} |\beta_{k-1}(t_k)^2\log(T^k(t_k))| \leq \sum_{k=0}^{K_y-1}\frac{1}{q_k^2}\log\left(\frac{1}{T^k(t_k)}\right) \leq \sum_{k=0}^{K_y-1}\frac{1}{q_k^2}\log\left(\frac{4}{T^k(x)}\right) \\
&\quad \leq \log(4)\sum_{k=0}^{\infty}\frac{1}{\mathrm{F}_{k+1}^2} + \sum_{k=0}^{K_y-1}\frac{1}{q_k^2}\log\left(\frac{2q_{k+1}}{q_k}\right) \leq \log(8)\sum_{k=0}^{\infty}\frac{1}{\mathrm{F}_{k+1}^2} + \log(q_{K_y})\sum_{k=0}^{\infty}\frac{1}{\mathrm{F}_{k+1}^2}.
\end{aligned}$$

We note that $y \in I_{K_y}(x)$ implies that $|x-y| \leq |I_{K_y}(x)| \leq \frac{1}{q_{K_y}^2}$, and hence $2\log(q_{K_y}) \leq \log\left(\frac{1}{|x-y|}\right)$. We have

$$\sum_{k=0}^{K_y-1} |u_{1,k}(x) - u_{1,k}(y)| \leq c_3|x-y|\log\left(\frac{1}{|x-y|}\right) + c_4|x-y|, \tag{3.2.2}$$

with $c_3 = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{\mathrm{F}_{k+1}^2}$ and $c_4 = (\log(16)+5)\sum_{k=0}^{\infty}\frac{1}{\mathrm{F}_{k+1}^2}$.

By the Mean Value Theorem and the same arguments as in the proof of Lemma 3.15, for some t_k between x and y we have

$$\begin{aligned}
&\sum_{k=0}^{K_y-1} |u_{2,k}(x) - u_{2,k}(y)| = \sum_{k=0}^{K_y-1} |x-y||(P(T^k(t_k)))'\beta_{k-1}(t_k)^2 - 4P(T^k(t_k))\beta_{k-1}(t_k)^3 q_{k-1}| \\
&\quad \leq |x-y|\Big(\sum_{k=0}^{K_y-1}\|P'\|_\infty\frac{1}{q_k^2} + 4\sum_{k=0}^{K_y-1}\|P\|_\infty\frac{1}{q_k^2}\Big) \leq (\|P'\|_\infty + 4\|P\|_\infty)|x-y|\sum_{k=0}^{\infty}\frac{1}{\mathrm{F}_{k+1}^2},
\end{aligned} \tag{3.2.3}$$

since $q_k(x) = q_k(t_k)$ for all $k \leq K_y$, for $\|P\|_\infty = \sup\limits_{y\in(0,1)}|P(y)|$ and $\|P'\|_\infty = \sup\limits_{y\in(0,1)}|P(y)'|$.

By the Mean Value Theorem and the same arguments as in the proof of Lemma 3.17, for some t_k between x and y we have

$$\begin{aligned}\sum_{k=0}^{K_y-1} |u_{3,k}(x) - u_{3,k}(y)| = \sum_{k=0}^{K_y-1} |x-y| \Big| \beta_k(t_k)^2 T^k(t_k) F_{2,3}(T^{k+1}(t_k)) \\ + \beta_{k-1}(t_k)^2 \int_0^{p(k)} t^2 F_{2,3}(T(t))dt + 4(-1)^{k+1}\mathcal{I}_k(t_k)\beta_{k-1}(t_k)^4 \sum_{j=0}^{k-1}(-1)^j \frac{T^j(t_k)}{\beta_j(t_k)^2}\Big| \\ \leq |x-y| \left(\sum_{k=0}^{K_y-1} \frac{1}{q_{k+1}^2}\|F_{2,3}\|_\infty + \sum_{k=0}^{K_y-1} \frac{1}{q_k^2}\|F_{2,3}\|_\infty + 4\sum_{k=0}^{K_y-1} \|F_{2,3}\|_\infty \frac{1}{q_k^2} \right) \\ \leq 6\|F_{2,3}\|_\infty |x-y| \sum_{k=0}^{\infty} \frac{1}{\mathrm{F}_{k+1}^2}. \quad (3.2.4)\end{aligned}$$

since $q_k(x) = q_k(t_k)$ for all $k \leq K_y$.

The result follows from (3.2.1)-(3.2.4) by letting $C_1 = 2c_1 + \frac{\pi^3}{6}\sum_{k=0}^{\infty} \frac{1}{\mathrm{F}_{k+1}^2}$ and $C_2 = 4\|F_{2,3}\|_\infty + c_2 + (\frac{\pi^3}{3}(\log(16)+5) + \|P'\|_\infty + 4\|P\|_\infty + 36\|F_{2,3}\|_\infty)\sum_{k=0}^{\infty}\frac{1}{\mathrm{F}_{k+1}^2}$. □

As we can see, we can choose the constants C_1, C_2 independent of x.

In the same way we show that for all $x \in (0,1) \setminus \mathbb{Q}$ and all $y \in (0,1)$ we have

$$|G_{2,3}(x) - G_{2,3}(y)| \leq C_3|x-y|\log\left(\frac{1}{|x-y|}\right) + C_4|x-y|,$$

for some constants $C_3, C_4 > 0$, which depend on x. The proof follows from (3.1.37), the arguments in the proof of Lemma 3.21 and Lemmas 3.22-3.24

3.3 Differentiability of $S_{3,2}$ and $T_{3,2}$

Let

$$\Upsilon_{d,s}(x) = \sum_{n_1=1}^{\infty} \dots \sum_{n_d=1}^{\infty} \frac{e^{i\pi(n_1^2+\dots+n_d^2)x}}{(n_1^2+\dots+n_d^2)^s},$$

whose imaginary part is $S_{d,s}$ and real part is $T_{d,s}$. We start by finding the functional equation for $\Upsilon_{3,2}$.

3.3.1 Functional equation for $\Upsilon_{3,2}$

We have the following proposition.

Proposition 3.35. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\theta = \left\langle \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle \subset SL_2(\mathbb{Z})$ with $c \neq 0$ and $x \in \mathbb{R}$. We have*

$$\begin{aligned}
\Upsilon_{3,2}(x) = \rho_\gamma^3 (cx+d)^{5/2} \Upsilon_{3,2}(\gamma \cdot x) + \frac{3\pi^2}{8c^2} \rho_\gamma^2 (cx+d) \mathrm{Log}(cx+d) \\
+ \tilde{A}(cx+d)^2 + \tilde{B}(cx+d)^{3/2} + \tilde{C}(cx+d) + \tilde{D}(cx+d)^{1/2} + \tilde{E} \\
+ \frac{3}{2}\rho_\gamma^2 \left(\rho_\gamma (cx+d)^{5/2} - (cx+d)^3\right)\left(\Upsilon_{2,2}(\gamma\cdot x) + \Upsilon_{1,2}(\gamma \cdot x)\right) \\
+ \frac{3}{4}\rho_\gamma \left((cx+d)^{7/2} - \rho_\gamma^2 (cx+d)^{5/2}\right) \Upsilon_{1,2}(\gamma \cdot x) \\
- \rho_\gamma^3 \int_{-\frac{d}{c}}^{x} \left(3c(ct+d)^{3/2} + \frac{3c^2}{4}(t-x)(ct+d)^{1/2}\right) \Upsilon_{3,2}(\gamma \cdot t)\, dt \\
- \frac{3c}{2}\rho_\gamma^2 \int_{-\frac{d}{c}}^{x} \Bigg(\Big(3\rho_\gamma (ct+d)^{3/2} - 4(ct+d)^2 + \Big(t-x\Big)\Big(\frac{3c}{4}\rho_\gamma (ct+d)^{1/2} - 2c(ct+d)\Big)\Big) \\
\left(\Upsilon_{2,2}(\gamma \cdot t) + \Upsilon_{1,2}(\gamma \cdot t)\right)\Bigg) dt \\
- \frac{3c}{4}\rho_\gamma \int_{-\frac{d}{c}}^{x} \Bigg(\Big(5(ct+d)^{5/2} - 3\rho_\gamma^2 (ct+d)^{3/2} + \Big(t-x\Big)\Big(\frac{15c}{4}(ct+d)^{3/2} - \frac{3c}{4}\rho_\gamma^2 (ct+d)^{1/2}\Big)\Big) \\
\Upsilon_{1,2}(\gamma \cdot t)\Bigg) dt
\end{aligned}$$

with

$$\begin{aligned}
\tilde{A} =& \frac{\pi^2}{16c^2} \\
\tilde{B} =& -\frac{\pi^2}{2c^2}\rho_\gamma \\
\tilde{C} =& -\frac{3\pi^2}{8c^2}\rho_\gamma^2 - \frac{\pi^2 d}{8c^2} + \frac{h_\gamma}{c} \\
\tilde{D} =& \frac{\rho_\gamma^3 \pi^2}{2c^2} \\
\tilde{E} =& \Upsilon_{3,2}\left(-\frac{d}{c}\right)
\end{aligned}$$

where Log *denotes the principal value of the complex logarithm, $x^{1/2}$ denote the principal value of the complex square root.*

The proof of Proposition 3.35 is very technical, therefore we will split the calculations into various lemmas and claims. Firstly, we note that $\Upsilon_{3,2}$ is differentiable in the upper-half plane, thus we have the following.

Claim 3.36. Let $z \in \mathbb{H}$. We have

$$\Upsilon_{3,2}'(z) = i\pi \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{e^{i\pi(n^2+m^2+k^2)z}}{n^2+m^2+k^2} = i\pi \Upsilon_{3,1}(z), \tag{3.3.1}$$

$$\Upsilon''_{3,2}(z) = -\pi^2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} e^{i\pi(n^2+m^2+k^2)z} = -\frac{\pi^2}{8}(\theta(z)-1)^3. \tag{3.3.2}$$

We then find a functional equation for $\Upsilon'_{3,2}$, which will be useful later.

Lemma 3.37. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_\theta$ with $c \neq 0$ and $\tau, \alpha \in \mathbb{H}$. We have*

$$\begin{aligned}
\Upsilon'_{3,2}(\tau) = \Upsilon'_{3,2}(\alpha) &+ \rho_\gamma^3(c\tau+d)^{1/2}\Upsilon'_{3,2}(\gamma\cdot\tau) - \rho_\gamma^3(c\alpha+d)^{1/2}\Upsilon'_{3,2}(\gamma\cdot\alpha) \\
&- \frac{c\rho_\gamma^3}{2}(c\tau+d)^{3/2}\Upsilon_{3,2}(\gamma\cdot\tau) + \frac{c\rho_\gamma^3}{2}(c\alpha+d)^{3/2}\Upsilon_{3,2}(\gamma\cdot\alpha) \\
&+ \rho_\gamma^3\int_\alpha^\tau \frac{3c^2}{4}(ct+d)^{1/2}\Upsilon_{3,2}(\gamma\cdot t)\,dt + \frac{3\pi^2}{8c}\rho_\gamma^2\mathrm{Log}(c\tau+d) \\
&- \frac{3\pi^2}{8c}\rho_\gamma^2\mathrm{Log}(c\alpha+d) \\
&- \frac{3\pi}{2i}\rho_\gamma^2\left(\rho_\gamma(c\tau+d)^{1/2} - (c\tau+d)\right)(\Upsilon_{2,1}(\gamma\cdot\tau)+\Upsilon_{1,1}(\gamma\cdot\tau)) \\
&+ \frac{3\pi}{2i}\rho_\gamma^2\left(\rho_\gamma(c\alpha+d)^{1/2} - (c\alpha+d)\right)(\Upsilon_{2,1}(\gamma\cdot\alpha)+\Upsilon_{1,1}(\gamma\cdot\alpha)) \\
&- \frac{3}{2}\rho_\gamma^2\left(\frac{c}{2}\rho_\gamma(c\tau+d)^{3/2} - c(c\tau+d)^2\right)(\Upsilon_{2,2}(\gamma\cdot\tau)+\Upsilon_{1,2}(\gamma\cdot\tau)) \\
&+ \frac{3}{2}\rho_\gamma^2\left(\frac{c}{2}\rho_\gamma(c\alpha+d)^{3/2} - c(c\alpha+d)^2\right)(\Upsilon_{2,2}(\gamma\cdot\alpha)+\Upsilon_{1,2}(\gamma\cdot\alpha)) \\
&+ \frac{3}{2}\rho_\gamma^2\int_\alpha^\tau\left(\frac{3c^2}{4}\rho_\gamma(ct+d)^{1/2} - 2c^2(ct+d)\right)(\Upsilon_{2,2}(\gamma\cdot t)+\Upsilon_{1,2}(\gamma\cdot t))dt \\
&- \frac{3\pi^2}{4c}\rho_\gamma(c\tau+d)^{1/2} + \frac{3\pi^2}{4c}\rho_\gamma(c\alpha+d)^{1/2} \\
&- \frac{3\pi}{4i}\rho_\gamma\left((c\tau+d)^{3/2} - \rho_\gamma^2(c\tau+d)^{1/2}\right)\Upsilon_{1,1}(\gamma\cdot\tau) \\
&+ \frac{3\pi}{4i}\rho_\gamma\left((c\alpha+d)^{3/2} - \rho_\gamma^2(c\alpha+d)^{1/2}\right)\Upsilon_{1,1}(\gamma\cdot\alpha) \\
&- \frac{3}{4}\rho_\gamma\left(\frac{3c}{2}(c\tau+d)^{5/2} - \frac{c}{2}\rho_\gamma^2(c\tau+d)^{3/2}\right)\Upsilon_{1,2}(\gamma\cdot\tau) \\
&+ \frac{3}{4}\rho_\gamma\left(\frac{3c}{2}(c\alpha+d)^{5/2} - \frac{c}{2}\rho_\gamma^2(c\alpha+d)^{3/2}\right)\Upsilon_{1,2}(\gamma\cdot\alpha) \\
&+ \frac{3}{4}\rho_\gamma\int_\alpha^\tau\left(\frac{15c^2}{4}(ct+d)^{3/2} - \frac{3c^2}{4}\rho_\gamma^2(ct+d)^{1/2}\right)\Upsilon_{1,2}(\gamma\cdot t)dt \\
&+ \frac{\rho_\gamma^3\pi^2}{4c}(c\tau+d)^{-1/2} - \frac{\rho_\gamma^3\pi^2}{4c}(c\alpha+d)^{-1/2} + \frac{\pi^2}{8}\tau - \frac{\pi^2}{8}\alpha.
\end{aligned}$$

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_\theta$. Fix $\alpha \in \mathbb{H}$. We have

$$\Upsilon'_{3,2}(x) = -\pi^2\int_{i\infty}^{\tau}\left(\frac{\theta(t)-1}{2}\right)^3 dt = -\pi^2\int_{i\infty}^{\alpha}\left(\frac{\theta(t)-1}{2}\right)^3 dt - \pi^2\int_{\alpha}^{\tau}\left(\frac{\theta(t)-1}{2}\right)^3 dt$$

$$= \Upsilon'_{3,2}(\alpha) - \frac{\pi^2}{8}\int_\alpha^\tau (\theta(t)-1)^3\,dt.$$

Then by (2.2.2) the second term becomes

$$\begin{aligned}
&-\frac{\pi^2}{8}\int_\alpha^\tau (\theta(t)-1)^3\,dt = -\frac{\pi^2}{8}\int_\alpha^\tau \left(\rho_\gamma(ct+d)^{-1/2}\theta(\gamma\cdot t)-1\right)^3 dt\\
&= \rho_\gamma^3\int_\alpha^\tau (ct+d)^{-3/2}\Upsilon''_3(\gamma\cdot t)\,dt - \frac{3\pi^2}{8}\rho_\gamma^2\int_\alpha^\tau \left(\rho_\gamma(ct+d)^{-3/2}-(ct+d)^{-1}\right)\theta(\gamma\cdot t)^2\,dt\\
&-\frac{3\pi^2}{8}\rho_\gamma\int_\alpha^\tau \left((ct+d)^{-1/2}-\rho_\gamma^2(ct+d)^{-3/2}\right)\theta(\gamma\cdot t)\,dt - \frac{\pi^2}{8}\int_\alpha^\tau \left(\rho_\gamma^3(ct+d)^{-3/2}-1\right)dt,
\end{aligned}$$

We evaluate each integral in turn. We have

$$\begin{aligned}
&\rho_\gamma^3\int_\alpha^\tau (ct+d)^{-3/2}\Upsilon''_3(\gamma\cdot t)\,dt\\
&\quad= \rho_\gamma^3(c\tau+d)^{1/2}\Upsilon'_{3,2}(\gamma\cdot\tau) - \rho_\gamma^3(c\alpha+d)^{1/2}\Upsilon'_{3,2}(\gamma\cdot\alpha) - \frac{c\rho_\gamma^3}{2}(c\tau+d)^{3/2}\Upsilon_{3,2}(\gamma\cdot\tau)\\
&\qquad+\frac{c\rho_\gamma^3}{2}(c\alpha+d)^{3/2}\Upsilon_{3,2}(\gamma\cdot\alpha) + \rho_\gamma^3\int_\alpha^\tau \frac{3c^2}{4}(ct+d)^{1/2}\Upsilon_{3,2}(\gamma\cdot t)\,dt.
\end{aligned}$$

Also, using the substitution $u=\gamma\cdot t$ and integrating by parts, we have

$$\begin{aligned}
&-\frac{3\pi^2}{8}\rho_\gamma^2\int_\alpha^\tau \left(\rho_\gamma(ct+d)^{-3/2}-(ct+d)^{-1}\right)\theta(\gamma\cdot t)^2\,dt\\
&= \frac{3\pi^2}{4c}\rho_\gamma^3(c\tau+d)^{-1/2} + \frac{3\pi^2}{8c}\rho_\gamma^2\mathrm{Log}(c\tau+d) - \frac{3\pi^2}{4c}\rho_\gamma^3(c\alpha+d)^{-1/2} - \frac{3\pi^2}{8c}\rho_\gamma^2\mathrm{Log}(c\alpha+d)\\
&\qquad-\frac{3\pi}{2i}\rho_\gamma^2\left(\rho_\gamma(c\tau+d)^{1/2}-(c\tau+d)\right)\left(\Upsilon_{2,1}(\gamma\cdot\tau)+\Upsilon_{1,1}(\gamma\cdot\tau)\right)\\
&\qquad+\frac{3\pi}{2i}\rho_\gamma^2\left(\rho_\gamma(c\alpha+d)^{1/2}-(c\alpha+d)\right)\left(\Upsilon_{2,1}(\gamma\cdot\alpha)+\Upsilon_{1,1}(\gamma\cdot\alpha)\right)\\
&\qquad-\frac{3}{2}\rho_\gamma^2\left(\frac{c}{2}\rho_\gamma(c\tau+d)^{3/2}-c(c\tau+d)^2\right)\left(\Upsilon_{2,2}(\gamma\cdot\tau)+\Upsilon_{1,2}(\gamma\cdot\tau)\right)\\
&\qquad+\frac{3}{2}\rho_\gamma^2\left(\frac{c}{2}\rho_\gamma(c\alpha+d)^{3/2}-c(c\alpha+d)^2\right)\left(\Upsilon_{2,2}(\gamma\cdot\alpha)+\Upsilon_{1,2}(\gamma\cdot\alpha)\right)\\
&\qquad+\frac{3}{2}\rho_\gamma^2\int_\alpha^\tau\left(\frac{3c^2}{4}\rho_\gamma(ct+d)^{1/2}-2c^2(ct+d)\right)\left(\Upsilon_{2,2}(\gamma\cdot t)+\Upsilon_{1,2}(\gamma\cdot t)\right)dt.
\end{aligned}$$

Again, using the substitution $u=\gamma\cdot t$ and integrating by parts, we obtain

$$\begin{aligned}
&-\frac{3\pi^2}{8}\rho_\gamma\int_\alpha^\tau \left((ct+d)^{-1/2}-\rho_\gamma^2(ct+d)^{-3/2}\right)\theta(\gamma\cdot t)\,dt\\
&= -\frac{3\pi^2}{4c}\rho_\gamma(c\tau+d)^{1/2} - \frac{3\pi^2}{4c}\rho_\gamma^3(c\tau+d)^{-1/2} + \frac{3\pi^2}{4c}\rho_\gamma(c\alpha+d)^{1/2} + \frac{3\pi^2}{4c}\rho_\gamma^3(c\alpha+d)^{-1/2}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{3\pi}{4i}\rho_\gamma \left((c\tau+d)^{3/2} - \rho_\gamma^2(c\tau+d)^{1/2}\right)\Upsilon_{1,1}(\gamma\cdot\tau)\\
&+ \frac{3\pi}{4i}\rho_\gamma \left((c\alpha+d)^{3/2} - \rho_\gamma^2(c\alpha+d)^{1/2}\right)\Upsilon_{1,1}(\gamma\cdot\alpha)\\
&- \frac{3}{4}\rho_\gamma \left(\frac{3c}{2}(c\tau+d)^{5/2} - \frac{c}{2}\rho_\gamma^2(c\tau+d)^{3/2}\right)\Upsilon_{1,2}(\gamma\cdot\tau)\\
&+ \frac{3}{4}\rho_\gamma \left(\frac{3c}{2}(c\alpha+d)^{5/2} - \frac{c}{2}\rho_\gamma^2(c\alpha+d)^{3/2}\right)\Upsilon_{1,2}(\gamma\cdot\alpha)\\
&+ \frac{3}{4}\rho_\gamma \int_\alpha^\tau \left(\frac{15c^2}{4}(ct+d)^{3/2} - \frac{3c^2}{4}\rho_\gamma^2(ct+d)^{1/2}\right)\Upsilon_{1,2}(\gamma\cdot t)dt.
\end{aligned}
$$

Finally, we have

$$
-\frac{\pi^2}{8}\int_\alpha^\tau \left(\rho_\gamma^3(ct+d)^{-3/2}-1\right)dt = \frac{\rho_\gamma^3\pi^2}{4c}(c\tau+d)^{-1/2} - \frac{\rho_\gamma^3\pi^2}{4c}(c\alpha+d)^{-1/2} + \frac{\pi^2}{8}\tau - \frac{\pi^2}{8}\alpha.
$$

Summing up gives the result. □

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_\theta$ define $h_\gamma : \mathbb{H} \to \mathbb{H}$ by

$$
\begin{aligned}
h_\gamma(z) = {}& \Upsilon_{3,2}'(z) - \rho_\gamma^3(cz+d)^{1/2}\Upsilon_{3,2}'(\gamma\cdot z) + \frac{c\rho_\gamma^3}{2}(cz+d)^{3/2}\Upsilon_{3,2}(\gamma\cdot z)\\
&+ \frac{3\pi^2}{4c}\rho_\gamma(cz+d)^{1/2} - \frac{\pi^2}{4c}\rho_\gamma^3(cz+d)^{-1/2} - \frac{3\pi^2}{8c}\rho_\gamma^2\mathrm{Log}(cz+d) - \frac{\pi^2}{8}z\\
&+ \frac{3\pi}{2i}\rho_\gamma^2\left(\rho_\gamma(cz+d)^{1/2} - (cz+d)\right)\left(\Upsilon_{2,1}(\gamma\cdot z)+\Upsilon_{1,1}(\gamma\cdot z)\right)\\
&+ \frac{3}{2}\rho_\gamma^2\left(\frac{c}{2}\rho_\gamma(cz+d)^{3/2} - c(cz+d)^2\right)\left(\Upsilon_{2,2}(\gamma\cdot z)+\Upsilon_{1,2}(\gamma\cdot z)\right)\\
&+ \frac{3\pi}{4i}\rho_\gamma\left((cz+d)^{3/2} - \rho_\gamma^2(cz+d)^{1/2}\right)\Upsilon_{1,1}(\gamma\cdot z)\\
&+ \frac{3}{4}\rho_\gamma\left(\frac{3c}{2}(cz+d)^{5/2} - \frac{c}{2}\rho_\gamma^2(cz+d)^{3/2}\right)\Upsilon_{1,2}(\gamma\cdot z)\\
&- \rho_\gamma^3\int_{-\frac{d}{c}}^z \frac{3c^2}{4}(ct+d)^{1/2}\Upsilon_{3,2}(\gamma\cdot t)\,dt\\
&- \frac{3}{2}\rho_\gamma^2\int_{-\frac{d}{c}}^z\left(\frac{3c^2}{4}\rho_\gamma(ct+d)^{1/2} - 2c^2(ct+d)\right)\left(\Upsilon_{2,2}(\gamma\cdot t)+\Upsilon_{1,2}(\gamma\cdot t)\right)dt\\
&- \frac{3}{4}\rho_\gamma\int_{-\frac{d}{c}}^z\left(\frac{15c^2}{4}(ct+d)^{3/2} - \frac{3c^2}{4}\rho_\gamma^2(ct+d)^{1/2}\right)\Upsilon_{1,2}(\gamma\cdot t)dt. \qquad (3.3.3)
\end{aligned}
$$

Since $\Upsilon_{3,2}$, $\Upsilon_{2,2}$, $\Upsilon_{1,2}$ are bounded on $\mathbb{C}$, the function h_γ is well defined.

Claim 3.38. For each $\gamma \in \Gamma_\theta$ the function h_γ is constant on $\mathbb{H}$. Moreover, if $\gamma_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c & d\end{smallmatrix}\right)$, $\gamma_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c & d\end{smallmatrix}\right) \in \Gamma_\theta$, then $h_{\gamma_1} = h_{\gamma_2}$.

Proof. By Lemma 3.37, for all $\alpha, \tau \in \mathbb{H}$ we have $h_\gamma(\alpha) = h_\gamma(\tau)$. Therefore, it is a constant function on $\mathbb{H}$. As seen in Section 2.2, ρ_γ depends only on c and d. Moreover, $\Upsilon_{i,j}$ is 2 periodic and by Remark 2.8 $\Upsilon_{i,j}(\gamma_1 \cdot z) = \Upsilon_{i,j}(\gamma_2 \cdot z)$ for $i \in \{1,2,3\}$ and $j \in \{1,2\}$. □

We can now prove Proposition 3.35.

Proof of Proposition 3.35. Fix $\alpha \in \mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\theta$ with $c \neq 0$, let $\tau \in \mathbb{H}$. Integrating by parts we get

$$\begin{aligned}
\Upsilon_{3,2}(\tau) =& \pi^2 \int_{i\infty}^{\tau} (t-\tau) \left(\frac{\theta(t)-1}{2}\right)^3 dt \\
=& \pi^2 \int_{i\infty}^{\alpha} (t-\tau) \left(\frac{\theta(t)-1}{2}\right)^3 dt + \pi^2 \int_{\alpha}^{\tau} (t-\tau) \left(\frac{\theta(t)-1}{2}\right)^3 dt \\
=& (\tau-\alpha)\Upsilon_{3,2}'(\alpha) + \Upsilon_{3,2}(\alpha) + \pi^2 \int_{\alpha}^{\tau} (t-\tau) \left(\frac{\theta(t)-1}{2}\right)^3 dt. \qquad (3.3.4)
\end{aligned}$$

We apply (2.2.2) to the second term, we then obtain:

$$\begin{aligned}
&\pi^2 \int_{\alpha}^{\tau} (t-\tau) \left(\frac{\theta(t)-1}{2}\right)^3 dt = \frac{\pi^2}{8} \int_{\alpha}^{\tau} (t-\tau) \left(\rho_\gamma (ct+d)^{-1/2} \theta(\gamma \cdot t) - 1\right)^3 dt \\
=& -\rho_\gamma^3 \int_{\alpha}^{\tau} (t-\tau)(ct+d)^{-3/2} \Upsilon_{3,2}''(\gamma \cdot t)\, dt \\
=& \frac{3\pi^2}{8} \rho_\gamma^2 \int_{\alpha}^{\tau} (t-\tau) \left(\rho_\gamma (ct+d)^{-3/2} - (ct+d)^{-1}\right) \theta(\gamma \cdot t))^2\, dt \\
&+ \frac{3\pi^2}{8} \rho_\gamma \int_{\alpha}^{\tau} (t-\tau) \left((ct+d)^{-1/2} - \rho_\gamma^2 (ct+d)^{-3/2}\right) \theta(\gamma \cdot t)\, dt \\
=& \frac{\pi^2}{8} \int_{\alpha}^{\tau} (t-\tau) \left(\rho_\gamma^3 (ct+d)^{-3/2} - 1\right) dt. \qquad (3.3.5)
\end{aligned}$$

We evaluate each integral in turn. We have

$$\begin{aligned}
-\rho_\gamma^3 &\int_{\alpha}^{\tau} (t-\tau)(ct+d)^{-3/2} \Upsilon_{3,2}''(\gamma \cdot t))\, dt \\
&= \rho_\gamma^3 (\alpha-\tau)(c\alpha+d)^{1/2} \Upsilon_{3,2}'(\gamma \cdot \alpha) + \rho_\gamma^3 (c\tau+d)^{5/2} \Upsilon_{3,2}(\gamma \cdot \tau) \\
&\quad - \rho_\gamma^3 \left((c\alpha+d)^{5/2} + \frac{c}{2}(\alpha-\tau)(c\alpha+d)^{3/2}\right) \Upsilon_{3,2}(\gamma \cdot \alpha) \\
&\quad - \rho_\gamma^3 \int_{\alpha}^{\tau} \left(3c(ct+d)^{3/2} + \frac{3c^2}{4}(t-\tau)(ct+d)^{1/2}\right) \Upsilon_{3,2}(\gamma \cdot t)\, dt. \qquad (3.3.6)
\end{aligned}$$

Also, using the substitution $u = \gamma \cdot t$ and integrating by parts we get:

$$\frac{3\pi^2}{8} \rho_\gamma^2 \int_{\alpha}^{\tau} (t-\tau) \left(\rho_\gamma (ct+d)^{-3/2} - (ct+d)^{-1}\right) \theta(\gamma \cdot t))^2\, dt$$

$$
\begin{aligned}
&= \frac{3\pi^2}{2c^2}\rho_\gamma^3(c\tau+d)^{1/2} + \frac{3\pi^2}{8c^2}\rho_\gamma^2(c\tau+d)\mathrm{Log}(c\tau+d) - \frac{3\pi^2}{8c^2}\rho_\gamma^2(c\tau+d) - \frac{3\pi^2}{4c^2}\rho_\gamma^3(c\tau+d)(c\alpha+d)^{-1/2} \\
&\quad - \frac{3\pi^2}{8c^2}\rho_\gamma^2(c\tau+d)\mathrm{Log}(c\alpha+d) - \frac{3\pi^2}{4c^2}\rho_\gamma^3(c\alpha+d)^{1/2} + \frac{3\pi^2}{8c^2}\rho_\gamma^2(c\alpha+d) \\
&\quad - \frac{3\pi}{2i}\rho_\gamma^2(\alpha-\tau)\left(\rho_\gamma(c\alpha+d)^{1/2} - (c\alpha+d)\right)\left(\Upsilon_{2,1}(\gamma\cdot\alpha) + \Upsilon_{1,1}(\gamma\cdot\alpha)\right) \\
&\quad + \frac{3}{2}\rho_\gamma^2\left(\rho_\gamma(c\tau+d)^{5/2} - (c\tau+d)^3\right)\left(\Upsilon_{2,2}(\gamma\cdot\tau) + \Upsilon_{1,2}(\gamma\cdot\tau)\right) \\
&\quad - \frac{3}{2}\rho_\gamma^2\left(\rho_\gamma(c\alpha+d)^{5/2} - (c\alpha+d)^3 + (\alpha-\tau)\left(\frac{c}{2}\rho_\gamma(c\alpha+d)^{3/2} - c(c\alpha+d)^2\right)\right) \\
&\qquad \cdot\left(\Upsilon_{2,1}(\gamma\cdot\alpha) + \Upsilon_{1,1}(\gamma\cdot\alpha)\right) \\
&\quad - \frac{3c}{2}\rho_\gamma^2\int_\alpha^\tau \left(\left(3\rho_\gamma(ct+d)^{3/2} - 4(ct+d)^2 + \Big(t-\tau\Big)\Big(\frac{3c}{4}\rho_\gamma(ct+d)^{1/2} - 2c(ct+d)\Big)\right)\right. \\
&\qquad \left.\cdot\left(\Upsilon_{2,2}(\gamma\cdot t) + \Upsilon_{1,2}(\gamma\cdot t)\right)\right)dt. \quad (3.3.7)
\end{aligned}
$$

Again, using the substitution $u = \gamma\cdot t$ and integrating by parts we get:

$$
\begin{aligned}
&\frac{3\pi^2}{8}\rho_\gamma\int_\alpha^\tau (t-\tau)\left((ct+d)^{-1/2} - \rho_\gamma^2(ct+d)^{-3/2}\right)\theta\left(\gamma\cdot t\right)\right)dt \\
&= -\frac{\pi^2}{2c^2}\rho_\gamma(c\tau+d)^{3/2} - \frac{3\pi^2}{2c^2}\rho_\gamma^3(c\tau+d)^{1/2} - \frac{\pi^2}{4c^2}\rho_\gamma(c\alpha+d)^{3/2} \\
&\quad + \frac{3\pi^2}{4c^2}\rho_\gamma^3(c\alpha+d)^{1/2} + \frac{3\pi^2}{4c^2}\rho_\gamma(c\tau+d)(c\alpha+d)^{1/2} + \frac{3\pi^2}{4c^2}(c\tau+d)\rho_\gamma^3(c\alpha+d)^{-1/2} \\
&\quad - \frac{3\pi}{4i}\rho_\gamma(\alpha-\tau)\left((c\alpha+d)^{3/2} - \rho_\gamma^2(c\alpha+d)^{1/2}\right)\Upsilon_{1,1}(\gamma\cdot\alpha) \\
&\quad + \frac{3}{4}\rho_\gamma\left((c\tau+d)^{7/2} - \rho_\gamma^2(c\tau+d)^{5/2}\right)\Upsilon_{1,2}(\gamma\cdot\tau) \\
&\quad - \frac{3}{4}\rho_\gamma\left((c\alpha+d)^{7/2} - \rho_\gamma^2(c\alpha+d)^{5/2} + (\alpha-\tau)\left(\frac{3c}{2}(c\alpha+d)^{5/2} - \frac{c}{2}\rho_\gamma^2(c\alpha+d)^{3/2}\right)\right) \\
&\qquad \cdot\Upsilon_{1,2}(\gamma\cdot\alpha) \\
&\quad - \frac{3c}{4}\rho_\gamma\int_\alpha^\tau \left(\left(5(ct+d)^{5/2} - 3\rho_\gamma^2(ct+d)^{3/2} + \Big(t-\tau\Big)\Big(\frac{15c}{4}(ct+d)^{3/2} - \frac{3c}{4}\rho_\gamma^2(ct+d)^{1/2}\Big)\right)\right. \\
&\qquad \left.\cdot\Upsilon_{1,2}(\gamma\cdot t)\right)dt. \quad (3.3.8)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\frac{\pi^2}{8}\int_\alpha^\tau (t-\tau)\left(\rho_\gamma^3(ct+d)^{-3/2} - 1\right)dt &= \frac{\rho_\gamma^3\pi^2}{2c^2}(c\tau+d)^{1/2} - \frac{\rho_\gamma^3\pi^2}{4c^2}(c\alpha+d)^{1/2} \\
&\quad - \frac{\rho_\gamma^3\pi^2}{4c^2}(c\tau+d)(c\alpha+d)^{-1/2} + \frac{\pi^2}{16}\tau^2 + \frac{\pi^2}{16}\alpha^2 - \frac{\pi^2}{8}\tau\alpha. \quad (3.3.9)
\end{aligned}
$$

Substituting (3.3.5)-(3.3.9) into (3.3.4) and gathering the terms, we get

$$
\begin{aligned}
\Upsilon_{3,2}(\tau) = {} & \rho_\gamma^3 (c\tau+d)^{5/2}\Upsilon_{3,2}\left(\gamma\cdot\tau\right) + \frac{3\pi^2}{8c^2}\rho_\gamma^2(c\tau+d)\mathrm{Log}(c\tau+d) \\
& + A\tau^2 + B(c\tau+d)^{3/2} + C(c\tau+d) + F\tau + D(c\tau+d)^{1/2} + E \\
& + \frac{3}{2}\rho_\gamma^2\left(\rho_\gamma(c\tau+d)^{5/2} - (c\tau+d)^3\right)\left(\Upsilon_{2,2}(\gamma\cdot\tau) + \Upsilon_{1,2}(\gamma\cdot\tau)\right) \\
& + \frac{3}{4}\rho_\gamma\left((c\tau+d)^{7/2} - \rho_\gamma^2(c\tau+d)^{5/2}\right)\Upsilon_{1,2}(\gamma\cdot\tau) \\
& - \rho_\gamma^3\int_\alpha^\tau \left(3c(ct+d)^{3/2} + \frac{3c^2}{4}\left(t-\tau\right)(ct+d)^{1/2}\right)\Upsilon_{3,2}\left(\gamma\cdot t\right)dt \\
& - \frac{3c}{2}\rho_\gamma^2\int_\alpha^\tau \Bigg(\Big(3\rho_\gamma(ct+d)^{3/2} - 4(ct+d)^2 + \Big(t-\tau\Big)\Big(\frac{3c}{4}\rho_\gamma(ct+d)^{1/2} - 2c(ct+d)\Big)\Big) \\
& \qquad \left(\Upsilon_{2,2}(\gamma\cdot t) + \Upsilon_{1,2}(\gamma\cdot t)\right)\Bigg)dt \\
& - \frac{3c}{4}\rho_\gamma\int_\alpha^\tau \Bigg(\Big(5(ct+d)^{5/2} - 3\rho_\gamma^2(ct+d)^{3/2} + \Big(t-\tau\Big)\Big(\frac{15c}{4}(ct+d)^{3/2} - \frac{3c}{4}\rho_\gamma^2(ct+d)^{1/2}\Big)\Big) \\
& \qquad \Upsilon_{1,2}(\gamma\cdot t)\Bigg)dt
\end{aligned}
$$

with

$$
\begin{aligned}
A = {} & \frac{\pi^2}{16} \\
B = {} & -\frac{\pi^2}{2c^2}\rho_\gamma \\
C = {} & -\frac{3\pi^2}{8c^2}\rho_\gamma^2 + \frac{3\pi^2}{4c^2}\rho_\gamma(c\alpha+d)^{1/2} - \frac{3\pi^2}{8c^2}\rho_\gamma^2\mathrm{Log}(c\alpha+d) - \frac{\pi^2}{4c^2}\rho_\gamma^3(c\alpha+d)^{-1/2} \\
F = {} & \rho_\gamma^3\frac{c}{2}(c\alpha+d)^{3/2}\Upsilon_{3,2}\left(\gamma\cdot\alpha\right) + \Upsilon'_{3,2}(\alpha) - \rho_\gamma^3(c\alpha+d)^{1/2}\Upsilon'_{3,2}\left(\gamma\cdot\alpha\right) - \frac{\pi^2}{8}\tau\alpha \\
& + \frac{3\pi}{2i}\rho_\gamma^2\left(\rho_\gamma(c\alpha+d)^{1/2} - (c\alpha+d)\right)\left(\Upsilon_{2,1}(\gamma\cdot\alpha) + \Upsilon_{1,1}(\gamma\cdot\alpha)\right) \\
& + \frac{3}{2}\rho_\gamma^2\left(\frac{c}{2}\rho_\gamma(c\alpha+d)^{3/2} - c(c\alpha+d)^2\right)\left(\Upsilon_{2,1}(\gamma\cdot\alpha) + \Upsilon_{1,1}(\gamma\cdot\alpha)\right) \\
& + \frac{3\pi}{4i}\rho_\gamma\left((c\alpha+d)^{3/2} - \rho_\gamma^2(c\alpha+d)^{1/2}\right)\Upsilon_{1,1}(\gamma\cdot\alpha) \\
& + \frac{3}{4}\rho_\gamma\left(\frac{3c}{2}(c\alpha+d)^{5/2} - \frac{c}{2}\rho_\gamma^2(c\alpha+d)^{3/2}\right)\Upsilon_{1,2}(\gamma\cdot\alpha) \\
= {} & h_\gamma + \frac{\rho_\gamma^3\pi^2}{4c}(c\alpha+d)^{-1/2} - \frac{3\pi^2}{4c}\rho_\gamma(c\alpha+d)^{1/2} + \frac{3\pi^2}{8c}\rho_\gamma^2\mathrm{Log}(c\alpha+d) \\
& + \rho_\gamma^3\int_{-\frac{d}{c}}^\alpha \frac{3c^2}{4}(ct+d)^{1/2}\Upsilon_{3,2}\left(\gamma\cdot t\right)dt \\
& + \frac{3}{2}\rho_\gamma^2\int_{-\frac{d}{c}}^\alpha \left(\frac{3c^2}{4}\rho_\gamma(ct+d)^{1/2} - 2c^2(ct+d)\right)\left(\Upsilon_{2,2}(\gamma\cdot t) + \Upsilon_{1,2}(\gamma\cdot t)\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{4}\rho_\gamma\int_{-\frac{d}{c}}^{\alpha}\left(\frac{15c^2}{4}(ct+d)^{3/2}-\frac{3c^2}{4}\rho_\gamma^2(ct+d)^{1/2}\right)\Upsilon_{1,2}(\gamma\cdot t)dt \qquad \text{by Lemma 3.37}\\
D=&\frac{\rho_\gamma^3\pi^2}{2c^2}\\
E=&\Upsilon_{3,2}(\alpha)-\rho_\gamma^3\left((c\alpha+d)^{5/2}+\frac{c}{2}\alpha(c\alpha+d)^{3/2}\right)\Upsilon_{3,2}(\gamma\cdot\alpha)-\alpha\Upsilon_{3,2}'(\alpha)\\
&+\rho_\gamma^3\alpha(c\alpha+d)^{1/2}\Upsilon_{3,2}'(\gamma\cdot\alpha)+\frac{\pi^2}{16}\alpha^2-\frac{\pi^2}{4c^2}\rho_\gamma(c\alpha+d)^{3/2}+\frac{3\pi^2}{8c^2}\rho_\gamma^2(c\alpha+d)\\
&-\frac{\rho_\gamma^3\pi^2}{4c^2}(c\alpha+d)^{1/2}-\frac{3\pi}{2i}\rho_\gamma^2\alpha\left(\rho_\gamma(c\alpha+d)^{1/2}-(c\alpha+d)\right)(\Upsilon_{2,1}(\gamma\cdot\alpha)+\Upsilon_{1,1}(\gamma\cdot\alpha))\\
&-\frac{3}{2}\rho_\gamma^2\left(\rho_\gamma(c\alpha+d)^{5/2}-(c\alpha+d)^3+\alpha\right)\left(\frac{c}{2}\rho_\gamma(c\alpha+d)^{3/2}-c(c\alpha+d)^2\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot(\Upsilon_{2,1}(\gamma\cdot\alpha)+\Upsilon_{1,1}(\gamma\cdot\alpha))\\
&-\frac{3\pi}{4i}\rho_\gamma\alpha\left((c\alpha+d)^{3/2}-\rho_\gamma^2(c\alpha+d)^{1/2}\right)\Upsilon_{1,1}(\gamma\cdot\alpha)\\
&-\frac{3}{4}\rho_\gamma\Big((c\alpha+d)^{7/2}-\rho_\gamma^2(c\alpha+d)^{5/2}+\alpha\Big(\frac{3c}{2}(c\alpha+d)^{5/2}-\frac{c}{2}\rho_\gamma^2(c\alpha+d)^{3/2}\Big)\Big)\Upsilon_{1,2}(\gamma\cdot\alpha)\\
=&\Upsilon_{3,2}(\alpha)-\rho_\gamma^3(c\alpha+d)^{5/2}\Upsilon_{3,2}(\gamma\cdot\alpha)-\frac{\pi^2}{16}\alpha^2-\frac{\pi^2}{4c^2}\rho_\gamma(c\alpha+d)^{3/2}-\alpha h_\gamma+\frac{3\pi^2}{8c^2}\rho_\gamma^2(c\alpha+d)\\
&-\frac{\pi^2}{4c^2}\rho_\gamma^3(c\alpha+d)^{1/2}+\alpha\frac{3\pi^2}{4c}\rho_\gamma(c\alpha+d)^{1/2}-\alpha\frac{\rho_\gamma^3\pi^2}{4c}(c\alpha+d)^{-1/2}-\alpha\frac{3\pi^2}{8c}\rho_\gamma^2\mathrm{Log}(c\alpha+d)\\
&-\frac{3}{2}\rho_\gamma^2\left(\rho_\gamma(c\alpha+d)^{5/2}-(c\alpha+d)^3\right)(\Upsilon_{2,2}(\gamma\cdot\alpha)+\Upsilon_{1,2}(\gamma\cdot\alpha)\\
&-\frac{3}{4}\rho_\gamma\left((c\alpha+d)^{7/2}-\rho_\gamma^2(c\alpha+d)^{5/2}\right)\Upsilon_{1,2}(\gamma\cdot\alpha)\\
&-\alpha\rho_\gamma^3\int_{-\frac{d}{c}}^{\alpha}\frac{3c^2}{4}(ct+d)^{1/2}\Upsilon_{3,2}(\gamma\cdot t)\,dt\\
&-\alpha\frac{3}{2}\rho_\gamma^2\int_{-\frac{d}{c}}^{\alpha}\left(\frac{3c^2}{4}\rho_\gamma(ct+d)^{1/2}-2c^2(ct+d)\right)(\Upsilon_{2,2}(\gamma\cdot t)+\Upsilon_{1,2}(\gamma\cdot t))dt\\
&-\alpha\frac{3}{4}\rho_\gamma\int_{-\frac{d}{c}}^{\alpha}\left(\frac{15c^2}{4}(ct+d)^{3/2}-\frac{3c^2}{4}\rho_\gamma^2(ct+d)^{1/2}\right)\Upsilon_{1,2}(\gamma\cdot t)dt \qquad \text{by Lemma 3.37.}
\end{aligned}
$$

Rearranging gives

$$
\begin{aligned}
\Upsilon_{3,2}(\tau)=&\rho_\gamma^3(c\tau+d)^{5/2}\Upsilon_{3,2}(\gamma\cdot\tau)+\frac{3\pi^2}{8c^2}\rho_\gamma^2(c\tau+d)\mathrm{Log}(c\tau+d)\\
&+\tilde{A}(c\tau+d)^2+\tilde{B}(c\tau+d)^{3/2}+\tilde{C}(c\tau+d)+\tilde{D}(c\tau+d)^{1/2}+\tilde{E}\\
&+\frac{3}{2}\rho_\gamma^2\left(\rho_\gamma(c\tau+d)^{5/2}-(c\tau+d)^3\right)(\Upsilon_{2,2}(\gamma\cdot\tau)+\Upsilon_{1,2}(\gamma\cdot\tau))\\
&+\frac{3}{4}\rho_\gamma\left((c\tau+d)^{7/2}-\rho_\gamma^2(c\tau+d)^{5/2}\right)\Upsilon_{1,2}(\gamma\cdot\tau)
\end{aligned}
$$

$$
\begin{aligned}
&- \rho_\gamma^3 \int_\alpha^\tau \left(3c(ct+d)^{3/2} + \frac{3c^2}{4}(t-\tau)(ct+d)^{1/2} \right) \Upsilon_{3,2}(\gamma \cdot t)\, dt \\
&- \frac{3c}{2}\rho_\gamma^2 \int_\alpha^\tau \Bigg(\Big(3\rho_\gamma (ct+d)^{3/2} - 4(ct+d)^2 + \Big(t-\tau\Big)\Big(\frac{3c}{4}\rho_\gamma (ct+d)^{1/2} - 2c(ct+d)\Big)\Big) \\
&\qquad (\Upsilon_{2,2}(\gamma \cdot t) + \Upsilon_{1,2}(\gamma \cdot t))\Bigg) dt \\
&- \frac{3c}{4}\rho_\gamma \int_\alpha^\tau \Bigg(\Big(5(ct+d)^{5/2} - 3\rho_\gamma^2 (ct+d)^{3/2} + \Big(t-\tau\Big)\Big(\frac{15c}{4}(ct+d)^{3/2} - \frac{3c}{4}\rho_\gamma^2 (ct+d)^{1/2}\Big)\Big) \\
&\qquad \Upsilon_{1,2}(\gamma \cdot t)\Bigg) dt \qquad (3.3.10)
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{A} =& \frac{\pi^2}{16} \\
\tilde{B} =& -\frac{\pi^2}{2c^2}\rho_\gamma \\
\tilde{C} =& -\frac{3\pi^2}{8c^2}\rho_\gamma^2 - \frac{\pi^2 d}{8c^2} + \frac{h_\gamma}{c} + \rho_\gamma^3 \int_{-\frac{d}{c}}^{\alpha} \frac{3c^2}{4}(ct+d)^{1/2}\eta_{3,2}(\gamma \cdot t)\, dt \\
&+ \frac{3}{2}\rho_\gamma^2 \int_{-\frac{d}{c}}^{\alpha} \left(\frac{3c^2}{4}\rho_\gamma (ct+d)^{1/2} - 2c^2(ct+d) \right) (\eta_{2,2}(\gamma \cdot t) + \eta_{1,2}(\gamma \cdot t)) dt \\
&+ \frac{3}{4}\rho_\gamma \int_{-\frac{d}{c}}^{\alpha} \left(\frac{15c^2}{4}(ct+d)^{3/2} - \frac{3c^2}{4}\rho_\gamma^2 (ct+d)^{1/2} \right) \eta_{1,2}(\gamma \cdot t) dt \\
\tilde{D} =& \frac{\rho_\gamma^3 \pi^2}{2c^2} \\
\tilde{E} =& \eta_{3,2}(\alpha) - \rho_\gamma^3 (c\alpha+d)^{5/2}\eta_{3,2}(\gamma \cdot \alpha) - \frac{\pi^2}{16c^2}(c\alpha+d)^2 - (c\alpha+d)h_\gamma + \frac{3\pi^2}{8c^2}\rho_\gamma^2 (c\alpha+d) \\
&- \frac{\pi^2}{2c^2}\rho_\gamma^3 (c\alpha+d)^{1/2} - \frac{3\pi^2}{8c}\rho_\gamma^2 (c\alpha+d)\mathrm{Log}(c\alpha+d) \\
&- \frac{3}{2}\rho_\gamma^2 \left(\rho_\gamma (c\alpha+d)^{5/2} - (c\alpha+d)^3 \right) (\eta_{2,2}(\gamma \cdot \alpha) + \eta_{1,2}(\gamma \cdot \alpha)) \\
&- \frac{3}{4}\rho_\gamma \left((c\alpha+d)^{7/2} - \rho_\gamma^2 (c\alpha+d)^{5/2} \right) \eta_{1,2}(\gamma \cdot \alpha) \\
&- (c\alpha+d)\rho_\gamma^3 \int_{-\frac{d}{c}}^{\alpha} \frac{3c}{4}(ct+d)^{1/2}\eta_{3,2}(\gamma \cdot t)\, dt \\
&- (c\alpha+d)\frac{3}{2}\rho_\gamma^2 \int_{-\frac{d}{c}}^{\alpha} \left(\frac{3c}{4}\rho_\gamma (ct+d)^{1/2} - 2c(ct+d) \right) (\eta_{2,2}(\gamma \cdot t) + \eta_{1,2}(\gamma \cdot t)) dt \\
&- (c\alpha+d)\frac{3}{4}\rho_\gamma \int_{-\frac{d}{c}}^{\alpha} \left(\frac{15c}{4}(ct+d)^{3/2} - \frac{3c}{4}\rho_\gamma^2 (ct+d)^{1/2} \right) \eta_{1,2}(\gamma \cdot t) dt.
\end{aligned}
$$

Then we observe that if we let $\alpha \to -\frac{d}{c}$, Equation 3.3.10 is well defined. Letting $\tau \to x \in \mathbb{R}$ gives the result. $\square$

3.3.2 Proof of Theorems 1.12 and 1.13

In this section, we prove Theorems 1.12 and 1.13. For the convenience of the reader, we recall them and then prove them simultaneously.

Theorem 1.12. *Neither $S_{3,2}$ nor $T_{3,2}$ is differentiable at 0.*

Theorem 1.13. *The functions $S_{3,2}$ and $T_{3,2}$ are not differentiable at any rational point $\frac{p}{q}$ such that p and q are not both odd. However, if $p \in 4\mathbb{Z}+3$, then $S_{3,2}$ is right differentiable, and if $p \in 4\mathbb{Z}+1$, then $S_{3,2}$ is left differentiable at $\frac{p}{q}$.*

Proof. Let $\frac{p}{q} \in \mathbb{Q}$, p, q coprime and not both odd, if $x = 0$, then let $q = 1$, $p = 0$. Since $\frac{p}{q}$ belongs to Γ_θ-orbit of 0 (see for example [Dui91]), we can choose $\gamma = \left(\begin{smallmatrix} a & b \\ q & -p \end{smallmatrix}\right) \in \Gamma_\theta$. We note that by the definition of ρ_γ, Remark 2.8 and Claim 3.38 the choice of γ does not matter. As $|\Upsilon_{3,2}|, |\Upsilon_{2,2}|, |\Upsilon_{1,2}|$ are bounded, Proposition 3.35 implies that

$$\begin{aligned}\Upsilon_{3,2}(x) = \Upsilon_{3,2}\left(\frac{p}{q}\right) + \frac{\rho_\gamma^3\pi^2}{2q^2}(qx-p)^{1/2} + \frac{3\pi^2}{8q^2}\rho_\gamma^2(qx-p)\mathrm{Log}(qx-p) \\ + \left(-\frac{3\pi^2}{8q^2}\rho_\gamma^2 + \frac{\pi^2 p}{8q^2} + \frac{h_\gamma}{q}\right)(qx-p) + O((qx-p)^{3/2}) \quad (3.3.11)\end{aligned}$$

as $x \to \frac{p}{q}$. We take the imaginary part of both sides of Equation (3.3.11), and we obtain that as $x \to \frac{p}{q}^+$ we have

$$\begin{aligned}S_{3,2}(x) = S_{3,2}\left(\frac{p}{q}\right) + \frac{\mathrm{Im}(\rho_\gamma^3)\pi^2}{2q^2}(qx-p)^{1/2} + \frac{3\pi^2}{8q^2}\mathrm{Im}(\rho_\gamma^2)(qx-p)\log(qx-p) \\ + \left(-\frac{3\pi^2}{8q^2}\mathrm{Im}(\rho_\gamma^2) + \frac{\mathrm{Im}(h_\gamma)}{q}\right)(qx-p) + O((qx-p)^{3/2}), \quad (3.3.12)\end{aligned}$$

and as $x \to \frac{p}{q}^-$ we have

$$\begin{aligned}S_{3,2}(x) = S_{3,2}\left(\frac{p}{q}\right) + \frac{\mathrm{Re}(\rho_\gamma^3)\pi^2}{2q^2}|qx-p|^{1/2} + \frac{3\pi^2}{8q^2}\mathrm{Im}(\rho_\gamma^2)(qx-p)\log|qx-p| \\ + \left(-\frac{3\pi^2}{8q^2}\mathrm{Im}(\rho_\gamma^2) + \frac{\mathrm{Im}(h_\gamma)}{q} + \frac{3\pi^2}{8q^2}\mathrm{Re}(\rho_\gamma^2)\right)(qx-p) + O(|qx-p|^{3/2}). \quad (3.3.13)\end{aligned}$$

By definition of ρ_γ, $\mathrm{Im}(\rho_\gamma^3)$ and $\mathrm{Im}(\rho_\gamma^2)$ are both 0, if $p \in 4\mathbb{Z}+3$, but then $\mathrm{Re}(\rho_\gamma^3)$ and $\mathrm{Im}(\rho_\gamma^2)$ are not both 0 (see Remark 2.7); therefore we deduce from (3.3.12) and (3.3.13) that $S_{3,2}$ is not differentiable at any $\frac{p}{q}$ with p, q not both odd. However, if $p \in 4\mathbb{Z}+3$ then $\mathrm{Im}(\rho_\gamma^3)$ and $\mathrm{Im}(\rho_\gamma^2)$ are both 0 and (3.3.12) implies that $S_{3,2}$ is right differentiable at $\frac{p}{q}$. On the other hand, if $p \in 4\mathbb{Z}+1$, then $\mathrm{Re}(\rho_\gamma^3)$ and $\mathrm{Im}(\rho_\gamma^2)$ are both 0 and (3.3.13) implies that $S_{3,2}$ is left differentiable at $\frac{p}{q}$.

We take the real part of both sides of Equation (3.3.11), and we obtain that as $x \to \frac{p}{q}^+$ we have

$$T_{3,2}(x) = T_{3,2}\left(\frac{p}{q}\right) + \frac{\mathrm{Re}(\rho_\gamma^3)\pi^2}{2q^2}(qx-p)^{1/2} + \frac{3\pi^2}{8q^2}\mathrm{Re}(\rho_\gamma^2)(qx-p)\log(qx-p) \\ + \left(-\frac{3\pi^2}{8q^2}\mathrm{Re}(\rho_\gamma^2) + \frac{\pi^2 p}{8q^2} + \frac{\mathrm{Re}(h_\gamma)}{q}\right)(qx-p) + O((qx-p)^{3/2}). \quad (3.3.14)$$

Since $\mathrm{Re}(\rho_\gamma^3)$ and $\mathrm{Re}(\rho_\gamma^2)$ cannot be both 0, we conclude from (3.3.14) that $T_{3,2}$ is not differentiable at $\frac{p}{q}$ with p, q not both odd. Also $x \to \frac{p}{q}^-$ we have

$$T_{3,2}(x) = T_{3,2}\left(\frac{p}{q}\right) - \frac{\mathrm{Im}(\rho_\gamma^3)\pi^2}{2q^2}|qx-p|^{1/2} + \frac{3\pi^2}{8q^2}\mathrm{Re}(\rho_\gamma^2)(qx-p)\log|qx-p| \\ + \left(-\frac{3\pi^2}{8q^2}\mathrm{Re}(\rho_\gamma^2) + \frac{\pi^2 p}{8q^2} + \frac{\mathrm{Re}(h_\gamma)}{q} - \frac{3\pi^2}{8q^2}\mathrm{Im}(\rho_\gamma^2)\right)(qx-p) + O(|qx-p|^{3/2}). \quad (3.3.15)$$

Since $\mathrm{Im}(\rho_\gamma^3)$ and $\mathrm{Re}(\rho_\gamma^2)$ cannot be both 0, we conclude from (3.3.15) that $T_{3,2}$ is neither left nor right differentiable at $\frac{p}{q}$. This completes the proof of the Theorems. $\square$

3.3.3 Proof of Theorem 1.14

In this section, we prove Theorem 1.14. For the convenience of the reader, we recall it.

Theorem 1.14. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $\mu_e(x) > 4$. Then neither $S_{3,2}$ nor $T_{3,2}$ is differentiable at x.*

Proof. We will prove the case of $T_{3,2}$, the case of $S_{3,2}$ is done in a very similar way. Let $x \in \mathbb{R} \setminus \mathbb{Q}$, such that there exists an infinite subsequence of continued fraction approximations of x, $\left(\frac{p_{n_k}}{q_{n_k}}\right)_k$, with p_{n_k}, q_{n_k} are not both odd and $\left|x - \frac{p_{n_k}}{q_{n_k}}\right| \leq \frac{1}{q_{n_k}^4}$. For each $\frac{p_{n_k}}{q_{n_k}}$, let $\gamma_k \in \Gamma_\theta$, such that $\gamma_k\left(\frac{p_{n_k}}{q_{n_k}}\right) = \infty$. Because of Claim 3.38 and the definition of ρ_γ we conclude that the choice of γ_k does not matter.

For brevity, let

$$\eta_\gamma(z) = \rho_\gamma^3(cz+d)^{5/2}\Upsilon_{3,2}(\gamma\cdot z) + \frac{\pi^2}{16c^2}(cz+d)^2 - \frac{\pi^2}{2c^2}\rho_\gamma(cz+d)^{3/2} \\ + \frac{3}{2}\rho_\gamma^2\left(\rho_\gamma(cz+d)^{5/2} - (cz+d)^3\right)\left(\Upsilon_{2,2}(\gamma\cdot z) + \Upsilon_{1,2}(\gamma\cdot z)\right) \\ + \frac{3}{4}\rho_\gamma\left((cz+d)^{7/2} - \rho_\gamma^2(cz+d)^{5/2}\right)\Upsilon_{1,2}(\gamma\cdot z) \\ - \rho_\gamma^3\int_{-\frac{d}{c}}^{z}\left(3c(ct+d)^{3/2} + \frac{3c^2}{4}(t-z)(ct+d)^{1/2}\right)\Upsilon_{3,2}(\gamma\cdot t)\,dt \\ - \frac{3c}{2}\rho_\gamma^2\int_{-\frac{d}{c}}^{z}\left(\left(3\rho_\gamma(ct+d)^{3/2} - 4(ct+d)^2\right.\right.$$

$$+\Big(t-z\Big)\Big(\frac{3c}{4}\rho_\gamma(ct+d)^{1/2}-2c(ct+d)\Big)\Big)(\Upsilon_{2,2}(\gamma\cdot t)+\Upsilon_{1,2}(\gamma\cdot t))\Big)dt$$
$$-\frac{3c}{4}\rho_\gamma\int_{-\frac{d}{c}}^{z}\Big(\Big(5(ct+d)^{5/2}-3\rho_\gamma^2(ct+d)^{3/2}$$
$$+\Big(t-z\Big)\Big(\frac{15c}{4}(ct+d)^{3/2}-\frac{3c}{4}\rho_\gamma^2(ct+d)^{1/2}\Big)\Big)\Upsilon_{1,2}(\gamma\cdot t)\Big)dt.$$

We observe that $\eta_\gamma\left(-\frac{d}{c}\right)=0$. With this notation, by Proposition 3.35 we have

$$\Upsilon_{3,2}(x)=\frac{3\pi^2}{8c^2}\rho_\gamma^2(cx+d)\mathrm{Log}(cx+d)+\left(-\frac{3\pi^2}{8c^2}\rho_\gamma^2-\frac{\pi^2 d}{8c^2}+\frac{h_\gamma}{c}\right)(cx+d)$$
$$+\frac{\rho_\gamma^3\pi^2}{2c^2}(cx+d)^{1/2}+\Upsilon_{3,2}\left(-\frac{d}{c}\right)+\eta_\gamma(x). \quad (3.3.16)$$

Then for all $z\in\mathbb{R}$ we have

$$\begin{aligned}|\eta_\gamma(z)|\leq&\Upsilon_{3,2}(0)|cz+d|^{5/2}+\frac{\pi^2}{16c^2}|cz+d|^2+\frac{\pi^2}{2c^2}|cz+d|^{3/2}\\
&+\frac{3}{2}\Upsilon_{2,2}(0)\left(|cz+d|^{5/2}+|cz+d|^3\right)\\
&+\frac{3}{4}\Upsilon_{1,2}(0)\left(|cz+d|^{7/2}+3|cz+d|^{5/2}+2|cz+d|^3\right)\\
&+\frac{15|c|}{4}\Upsilon_{3,2}(0)\left|\int_{-\frac{d}{c}}^{z}|ct+d|^{3/2}dt\right|+\frac{3|c|}{4}\Upsilon_{3,2}(0)|cz+d|\left|\int_{-\frac{d}{c}}^{z}|ct+d|^{1/2}dt\right|\\
&+\frac{3|c|}{2}(\Upsilon_{2,2}(0)+\Upsilon_{1,2}(0))\left|\int_{-\frac{d}{c}}^{z}\Big(6|ct+d|^2+\frac{15}{4}|ct+d|^{3/2}\Big)dt\right|\\
&+\frac{3|c|}{2}(\Upsilon_{2,2}(0)+\Upsilon_{1,2}(0))|cz+d|\left|\int_{-\frac{d}{c}}^{z}\Big(\frac{3}{4}|ct+d|^{1/2}+2|ct+d|\Big)dt\right|\\
&+\frac{15|c|}{16}\Upsilon_{1,2}(0)\left|\int_{-\frac{d}{c}}^{z}\Big(7|ct+d|^{5/2}+3|ct+d|^{3/2}\Big)dt\right|\\
&+\frac{9|c|}{16}\Upsilon_{1,2}(0)|cz+d|\int_{-\frac{d}{c}}^{z}\Big(5|ct+d|^{3/2}+|ct+d|^{1/2}\Big)dt\\
\leq&|cz+d|^{3/2}\Bigg(\Upsilon_{3,2}(0)|cz+d|+\frac{\pi^2}{16c^2}|cz+d|^{1/2}+\frac{\pi^2}{2c^2}\\
&+\frac{3}{2}\Upsilon_{2,2}(0)\left(|cz+d|+|cz+d|^{3/2}\right)+\frac{3}{4}\Upsilon_{1,2}(0)\left(|cz+d|^2+3|cz+d|+2|cz+d|^{3/2}\right)\\
&+2\Upsilon_{3,2}(0)|cz+d|+\frac{3}{2}(\Upsilon_{2,2}(0)+\Upsilon_{1,2}(0))\Big(3|cz+d|^{3/2}+2|cz+d|\Big)\\
&+\frac{3}{2}\Upsilon_{1,2}(0)\Big(2|ct+d|^2+|ct+d|\Big)\Bigg).\end{aligned}$$

Assume that $|cz+d|<1$, then we have

$$|\eta_\gamma(z)| \leq c_1|cz+d|^{3/2}, \tag{3.3.17}$$

for a constant $c_1 \in \mathbb{R}$.

Then we let y be such that:

1. $|q_{n_k}y - p_{n_k}| < 1$;
2. if $m\left(\frac{p_{n_k}}{q_{n_k}}\right) = 2 \mod 4$, then $y < \frac{p_{n_k}}{q_{n_k}}$, if $m\left(\frac{p_{n_k}}{q_{n_k}}\right) = 0 \mod 4$, then we will consider $y > \frac{p_{n_k}}{q_{n_k}}$ (this ensures that $\mathrm{Re}(\rho^3_{\gamma_k}(q_{n_k}y-p_{n_k})^{1/2}) \neq 0$);
3. $$\left|y - \frac{p_{n_k}}{q_{n_k}}\right| = a\left|x - \frac{p_{n_k}}{q_{n_k}}\right| \tag{3.3.18}$$
   for some constant a that will be described below.

Taking the real part of (3.3.16) we get

$$\begin{aligned}\left|T_{3,2}(y) - T_{3,2}\left(\frac{p_{n_k}}{q_{n_k}}\right)\right| &= \left|\frac{3\pi^2}{8q^2_{n_k}}\mathrm{Re}(\rho^2_{\gamma_k}(q_{n_k}y-p_{n_k})\log|q_{n_k}y-p_{n_k})|\right.\\ &\quad+\left(-\frac{3\pi^2}{8q^2_{n_k}}\mathrm{Re}(\rho^2_{\gamma_k}) - \frac{-\pi^2 p_{n_k}}{8q^2_{n_k}} + \frac{\mathrm{Re}(h_{\gamma_k})}{q_{n_k}}\right)(q_{n_k}y-p_{n_k})\\ &\quad\left.+\mathrm{Re}\left(\frac{\pi^2\rho^3_{\gamma_k}}{2q^2_{n_k}}(q_{n_k}y-p_{n_k})^{1/2}\right) + \mathrm{Re}(\eta(y))\right|\\ &\geq \frac{\pi^2}{2\sqrt{2}q^2_{n_k}}|q_{n_k}y-p_{n_k}|^{1/2} - \frac{3\pi^2}{8q^2_{n_k}}|q_{n_k}y-p_{n_k}|\log\left(\frac{1}{|q_{n_k}y-p_{n_k}|}\right)\\ &\quad-\left(\frac{3\pi^2}{8q_{n_k}} + \frac{\pi^2 p_{n_k}}{8q_{n_k}} + |h_{\gamma_k}| + \frac{3\pi^3}{8q_{n_k}}\right)\frac{1}{q_{n_k}}|q_{n_k}y-p_{n_k}| - c_1|q_{n_k}y-p_{n_k}|^{3/2}.\end{aligned}$$

We now consider $|h_{\gamma_k}|$. As it is a constant which depends on the pole of γ_k, we can estimate its size using (3.3.3) with $z = \frac{p_{n_k}}{q_{n_k}} + \frac{i}{q^\alpha_{n_k}}$. Then $\mathrm{Im}(z) = \frac{1}{q^\alpha_{n_k}}$, and $\mathrm{Im}(\gamma_k\cdot z) = \frac{1}{q^{2-\alpha}_{n_k}}$.

Let $z\in\mathbb{H}$. We first note that $\Upsilon_{2,1}(z) = \sum_{\ell=1}^\infty \frac{R_2(\ell)}{\ell}e^{i\pi\ell z}$, where $R_2(\ell)$ is the number of ways of writing ℓ as a sum of two squares where zeros are not allowed and the order matters. We have $R_2(\ell)\leq r_2(\ell)$, where $r_2(\ell)$ is the well-known sum of squares function, namely the number of ways of writing ℓ as a sum of two squares where zeros are allowed and the order and the sign matters. We have that $r_2(\ell) = \sum_{\substack{d|\ell\\ d\text{ odd}}}(-1)^{(d-1)/2}$, see for example [Ber06, p. 56]. It follows that $R_2(\ell) = o(\ell^\varepsilon)$ for all $\varepsilon>0$, therefore we conclude that $|\Upsilon_{2,1}(z)| \leq \frac{c_4}{\mathrm{Im}(z)}$, for some constant $c_2>0$ for all $z\in\mathbb{H}$. This bound is not optimal, nonetheless because of estimation of $|\Upsilon'_{3,2}(z)|$ which we will see in the next paragraph, it is sufficient in our case.

Also, $\Upsilon'_{3,2}(z) = \frac{1}{\pi}\Upsilon_{3,1}(z) = \frac{1}{\pi}\sum_{\ell=1}^{\infty}\frac{R_3(\ell)}{\ell}e^{i\pi\ell z}$, where $R_3(\ell)$ the sum of squares function, that is the number of ways of writing ℓ as a sum of three squares where zeros are not allowed and the order matters. We have $R_3(\ell) \le \sum_{n=1}^{\ell} r_2(n) = \pi\ell + O(\ell^{1/2}) = O(\ell)$, see [Har99, p. 67]. Therefore $|\Upsilon'_{3,2}(z)| \le c_3\sum_{\ell=1}^{\infty} e^{-\mathrm{Im}(z)\pi\ell} \le \frac{c_4}{\mathrm{Im}(z)}$, for some constant $c_4 > 0$ for all $z \in \mathbb{H}$.

Therefore, we have

$$\begin{aligned}|h_{\gamma_k}(z)| \le{}& \frac{9}{4}\Upsilon_{1,2}(0)q_{n_k}^{7/2-5/2\alpha} + \left(c_4 + \frac{3\pi c_4}{2} + 3\Upsilon_{2,2}(0) + 3\Upsilon_{1,2}(0)\right)q_{n_k}^{3-2\alpha}\\ &+\left(\frac{3\pi c_2}{2} + \Upsilon_{3,2}(0) + \frac{3}{2}\Upsilon_{2,2}(0) + \frac{9}{4}\Upsilon_{1,2}(0)\right)q_{n_k}^{5/2-3/2\alpha} + \frac{3\pi}{4}\Upsilon_{1,1}(0)q_{n_k}^{3/2-3/2\alpha}\\ &+\frac{3\pi}{2}\Upsilon_{1,1}(0)q_{n_k}^{1-\alpha} + \frac{9\pi}{4}\Upsilon_{1,1}(0)q_{n_k}^{1/2-1/2\alpha}\\ &+c_4 q_{n_k}^{\alpha} + \frac{3\pi^2}{4}q_{n_k}^{-1/2-1/2\alpha} + \frac{\pi^2}{4}q_{n_k}^{-3/2+1/2\alpha} + \frac{3\pi^2|1-\alpha|}{8q_{n_k}}\log(q_{n_k}) + \frac{\pi^2}{8}z.\end{aligned}$$

If we let $\alpha = 1$, noting that $|z|$ is bounded, we get

$$|h_{\gamma_k}(z)| \le c_5 q_{n_k},$$

for some constant c_5 independent of k and z. It follows that $\left|\frac{3\pi^2}{8q_{n_k}} + \frac{\pi^2 p_{n_k}}{8q_{n_k}} + |h_{\gamma_k}| + \frac{3\pi^3}{8q_{n_k}}\right| \le c_6 q_{n_k}$, for some constant c_6. Also observing that if $t > 1$, then $\log(t) \le 4t^{1/4}$ we have

$$\begin{aligned}\left|T_{3,2}(y) - T_{3,2}\left(\frac{p_{n_k}}{q_{n_k}}\right)\right| \ge{}& \frac{\pi^2}{2\sqrt{2}q_{n_k}^2}|q_{n_k}y - p_{n_k}|^{1/2} - \frac{3\pi^2}{2q_{n_k}^2}|q_{n_k}y - p_{n_k}|^{3/4}\\ &- c_6|q_{n_k}y - p_{n_k}| - c_1|q_{n_k}y - p_{n_k}|^{3/2}.\end{aligned}$$

By the choice of y we have

$$\begin{aligned}\left|T_{3,2}(y) - T_{3,2}\left(\frac{p_{n_k}}{q_{n_k}}\right)\right| \ge{}& \frac{\pi^2}{2\sqrt{2}q_{n_k}^2}a^{1/2}|q_{n_k}x - p_{n_k}|^{1/2} - \frac{3\pi^2}{2q_{n_k}^2}a^{3/4}|q_{n_k}x - p_{n_k}|^{3/4}\\ &- c_6 a|q_{n_k}x - p_{n_k}| - c_1 a^{3/2}|q_{n_k}x - p_{n_k}|^{3/2}\\ ={}& \frac{a^{1/2}}{q_{n_k}^2}|q_{n_k}x - p_{n_k}|^{1/2}\Big(\frac{\pi^2}{2\sqrt{2}} - \frac{3\pi^2}{2}|a^{1/4}q_{n_k}x - p_{n_k}|^{1/4}\\ &- c_6 a^{1/2}q_{n_k}^2|q_{n_k}x - p_{n_k}|^{1/2} - c_1 q_{n_k}^2 a|q_{n_k}x - p_{n_k}|\Big)\\ \ge{}& \frac{a^{1/2}}{q_{n_k}^2}|q_{n_k}x - p_{n_k}|^{1/2}\Big(\frac{\pi^2}{2\sqrt{2}} - \frac{3\pi^2}{2}a^{1/4} - c_6 a^{1/2} - c_1 a\Big).\end{aligned}$$

Now if we choose $a > 0$ such that $\frac{\pi^2}{2\sqrt{2}} - \frac{3\pi^2}{2}a^{1/4} - c_6 a^{1/2} - c_1 a > 0$, we have

$$\left|T_{3,2}(y) - T_{3,2}\left(\frac{p_{n_k}}{q_{n_k}}\right)\right| \ge \frac{b}{q_{n_k}^2}|q_{n_k}x - p_{n_k}|^{1/2},$$

for some constant $b > 0$, which is independent of k.

If $\left|T_{3,2}(x) - T_{3,2}\left(\frac{p_{n_k}}{q_{n_k}}\right)\right| \leq |T_{3,2}(x) - T_{3,2}(y)|$, then let $x_k = y$, otherwise let $x_k = \frac{p_{n_k}}{q_{n_k}}$. In either case $\left|T_{3,2}(y) - T_{3,2}\left(\frac{p_{n_k}}{q_{n_k}}\right)\right| \leq 2\,|T_{3,2}(x) - T_{3,2}(x_k)|$, and

$$|T_{3,2}(x) - T_{3,2}(x_k)| \geq \frac{C_1}{q_{n_k}^2}|q_{n_k}x - p_{n_k}|^{1/2},$$

with $C_1 = \frac{b}{2}$. Then we observe that

$$|x - x_k| \leq C_2\left|x - \frac{p_{n_k}}{q_{n_k}}\right|,$$

with $C_2 = a + 1$.We then have

$$\frac{|T_{3,2}(x) - T_{3,2}(x_k)|}{|x - x_k|} \geq \frac{\frac{C_1}{q_{n_k}^2}|q_{n_k}x - p_{n_k}|^{1/2}}{|x - x_k|} \geq \frac{C_1}{C_2 q_{n_k}^{3/2}\left|x - \frac{p_{n_k}}{q_{n_k}}\right|^{1/2}} \geq \frac{C_1}{C_2}q_{n_k}^{1/2} \to \infty,$$

as $k \to \infty$. This completes the proof of the the Theorem. □

Chapter 4

Jaffard's method

In this chapter we describe the relationship between wavelets and regularity, and we apply this method to $M_{k,s}, N_{k,s}, S_{d,s}$ and $T_{d,s}$ series. The first two sections correspond to the paper [Pet13].

4.1 Wavelet transform and regularity

For background information about wavelets, we refer the reader to the book by Ingrid Daubechies, "Ten Lectures on Wavelets" [Dau92], chapter 2 is especially relevant for this chapter.

We define the transform of an L^∞ function f with respect to the wavelet $\psi \in L^1(\mathbb{R})$ as follows:

$$C(a,b)(f) = \frac{1}{a}\int_{\mathbb{R}} f(t)\overline{\psi}\left(\frac{t-b}{a}\right)dt,$$

where $\overline{\psi}$ denotes the complex conjugate of ψ, $a > 0$, and $b \in \mathbb{R}$. On the other hand, we can reconstruct the function from its wavelet transform, using the formula:

$$f(t) = \int_0^\infty \frac{da}{a^2}\int_{\mathbb{R}} g\left(\frac{t-b}{a}\right)C(a,b)(f)db,$$

where g is a reconstruction wavelet, i.e. g satisfies the following:

a) $a^{-1}\widehat{\psi}(a)\widehat{g}(a) \in L^1(\mathbb{R})$;

b) $\int_0^\infty \overline{\widehat{\psi}}(a)\widehat{g}(a)\frac{da}{a} = 1$;

c) $\int_0^\infty \overline{\widehat{\psi}}(-a)\widehat{g}(-a)\frac{da}{a} = 1$;

It is not unique, but in some cases we can have $g = \psi$, see [HT91, p.160]. In the last 20 years, it has been established that wavelets, which originate from applied mathematics, can be very useful in the analysis of pointwise regularity. Apart from the paper by Holschneider and Tchmitchian [HT91], we should mention monographs by Stéphane Jaffard and Yves Meyer [JM96], [Mey98] in which they describe in detail the connection between wavelets and regularity. Also Oppenheim in his thesis [Opp97] applied wavelet theory in his study of regularity of a two-dimensional analogue of Riemann series (1.1.1).

We will denote the Fourier Transform of a function g by $\widehat{g}(\xi) = \int_{\mathbb{R}} g(x)e^{-ix\xi}dx$. We now recall Proposition 1 from [Jaf95].

Proposition J. *Let $\alpha > 0$, and $m = [\alpha]$ its integer part. Assume the following:*

1. $|\psi(x)| + |\psi^{(1)}(x)| + ... + |\psi^{(m+1)}(x)| \leq \frac{c}{(1+|x|)^{m+2}}$, *for some constant c;*
2. $\int_{\mathbb{R}} \psi(x)dx = \int_{\mathbb{R}} x\psi(x)dx = ... = \int_{\mathbb{R}} x^m \psi(x)dx = 0$;
3. $\widehat{\psi}(\xi) = 0$ *if* $\xi < 0$;
4. $\int_0^\infty |\widehat{\psi}(\xi)|^2 \frac{d\xi}{\xi} < \infty$.

Let $a \in (0,1)$, $b \in \mathbb{R}$. If $f : \mathbb{R} \to \mathbb{R} \in C^\alpha(x_0)$, then for some C that depends on x_0 and f, we have

$$|C(a,b)(f)| \leq Ca^\alpha \left(1 + \frac{|b - x_0|}{a}\right)^\alpha,$$

as $b \to x_0$ and $a \to 0$.

Conversely, if for some C that depends on x_0 and f we have

$$|C(a,b)(f)| \leq Ca^\alpha \left(1 + \frac{|b - x_0|}{a}\right)^{\alpha'} \text{ for an } \alpha' < \alpha,$$

as $b \to x_0$ and $a \to 0$, then $f \in C^\alpha(x_0)$.

4.1.1 The wavelet ψ_s

As in the previous chapter, we work with the principal branch $-\pi < \arg(z) \leq \pi$ of $z \in \mathbb{C}$. For $s > 0$ and $x \in \mathbb{R}$, consider

$$\psi_s(x) = \frac{1}{(x+i)^{s+1}}.$$

We now show that ψ_s satisfy the assumptions 2-4 of Proposition J. Assumption 1 will be considered separately in the proofs of Theorems 1.6 and 1.15. We start by noting the following fact that will be used later.

Lemma 4.1. *Let $\rho > 0$, $z \in \mathbb{C} \setminus \mathbb{R}$. We have*

$$\int_{\mathbb{R}} \frac{e^{it}}{(t-z)^\rho} dt = \begin{cases} \widetilde{c}(\rho)e^{iz} & \textit{if} \quad \mathrm{Im}(z) > 0, \\ 0 & \textit{if} \quad \mathrm{Im}(z) < 0, \end{cases}$$

with $\widetilde{c}(\rho) = \frac{2\pi e^{i\pi\rho/2}}{\Gamma(\rho)}$.

Proof. First assume that $\mathrm{Im}(z) > 0$. Then we have $\int_{\mathbb{R}} \frac{e^{it}}{(t-z)^\rho} dt = i^\rho \int_{\mathbb{R}} \frac{e^{it}}{(-iz+it)^\rho} dt$. The result follows from [GR07, p. 347] Equation 6, with $p = -1$, $\nu = \rho$ and $\beta = -iz$. On the other hand, if $\mathrm{Im}(z) < 0$, then $\int_{\mathbb{R}} \frac{e^{it}}{(t-z)^\rho} dt = e^{-i\pi\rho} i^\rho \int_{\mathbb{R}} \frac{e^{it}}{(iz-it)^\rho} dt$. The result then follows from [GR07, p. 347] Equation 7, with $p = -1$, $\nu = \rho$ and $\beta = iz$. However in this reference, no proof is given and therefore we provide a proof in Section 4.1.2 for the convenience of the reader. $\square$

Then we calculate the Fourier transform of ψ_s.

Lemma 4.2. *For $s > 1$, we have*

$$\widehat{\psi}_s(\xi) = \begin{cases} e^{-i\pi(s+1)}\xi^s \widetilde{c}(s+1)e^{-\xi} & \text{if} \quad \xi > 0, \\ 0 & \text{if} \quad \xi < 0. \end{cases}$$

Proof. By definition of the Fourier Transform, we have

$$\widehat{\psi}_s(\xi) = \int_{\mathbb{R}} \frac{e^{-ix\xi}}{(x+i)^{s+1}} dx = \begin{cases} e^{-i\pi(s+1)}\xi^s \int_{\mathbb{R}} \frac{e^{it}}{(t-i\xi)^{s+1}} dt & \text{if} \quad \xi > 0, \\ -e^{i\pi(s+1)}\xi^s \int_{\mathbb{R}} \frac{e^{it}}{(t-i\xi)^{s+1}} dt & \text{if} \quad \xi < 0. \end{cases}$$

We conclude by Lemma 4.1. □

• *Assumption 2*

Lemma 4.3. *For $s > 1$ and $\alpha < s$, we have*

$$\int_{\mathbb{R}} \psi_s(x)dx = \int_{\mathbb{R}} x\psi_s(x)dx = ... = \int_{\mathbb{R}} x^m \psi_s(x)dx = 0,$$

with $m = [\alpha]$.

Proof. Set $\widehat{\psi}_s(0) = 0$. Since $s > 0$, by Lemma 4.2 $\widehat{\psi}_s$ is a continuous function. Then for all $n < s$, we have

$$\widehat{\psi}_s^{(n)}(\xi) = \int_{\mathbb{R}} \frac{(-ix)^n e^{-ix\xi}}{(x+i)^{s+1}} dx = \begin{cases} \widetilde{c}(n,s)(\xi^s e^{-\xi})^{(n)} & \text{if} \quad \mathrm{Im}(i\xi) \geq 0, \\ 0 & \text{if} \quad \mathrm{Im}(i\xi) \leq 0, \end{cases}$$

for some constant $\widetilde{c}(n,s)$. In particular, the function is 0 at $\xi = 0$. As $\alpha < s$, it follows that for all $n \leq m$, we have

$$\int_{\mathbb{R}} \frac{x^n}{(x+i)^{s+1}} dx = 0.$$

□

• *Assumption 3*

See Lemma 4.2.

• *Assumption 4*

Lemma 4.4. *We have*

$$\int_0^\infty |\widehat{\psi}_s(\xi)|^2 \frac{d\xi}{\xi} < \infty.$$

Proof. By Lemma 4.2 we have

$$\int_0^\infty |\widehat{\psi}_s(\xi)|^2 \frac{d\xi}{\xi} = |\widetilde{c}(s+1)|^2 \int_0^\infty \xi^{2s-1} e^{-2\xi} d\xi = |\widetilde{c}(s+1)|^2 2^{-2s}\Gamma(2s) < \infty.$$

□

We have shown that ψ_s fulfils the assumptions 2-4 of Proposition J.

4.1.2 Proof of Lemma 4.1

In this section we present the proof of Lemma 4.1. In fact, we will prove something stronger, namely, we will show that the formula is true for all $\rho > 0$. First we observe, that if $\rho > 1$, then the integral $\int_{-\infty}^{\infty} \frac{e^{it}}{(t-z)^\rho} dt$ converges in the sense of Lebesgue. Otherwise, the integral is a convergent generalised Riemann integral. We divide the proof into three claims. First we consider $\rho \in (0,1)$, then $\rho \in \mathbb{R}^+ \setminus \mathbb{N}$, and finally $\rho \in \mathbb{N}$.

Claim 4.5. Let $\rho \in (0,1)$, and z such that $\mathrm{Im}(z) \neq 0$. Then

$$\int_{-\infty}^{\infty} \frac{e^{it}}{(t-z)^\rho} dt = \begin{cases} \frac{2\pi e^{i\pi\rho/2}}{\Gamma(\rho)} e^{iz} & \text{if} \quad \mathrm{Im}(z) > 0, \\ 0 & \text{if} \quad \mathrm{Im}(z) < 0. \end{cases}$$

Proof. We will consider two cases depending on the sign of $\mathrm{Im}(z)$.
Case 1: $\mathrm{Im}(z) < 0$**.** We fix $R > 0$ and we integrate along the contour Γ_R presented in Figure 4.1.

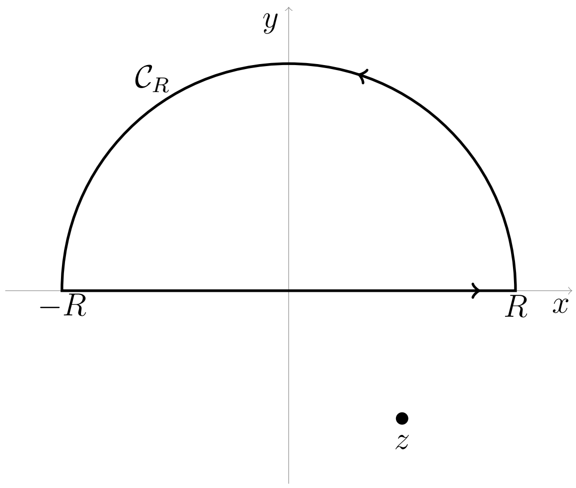

Figure 4.1: Contour Γ_R

Since $\mathrm{Im}(z) < 0$, the function $t \mapsto \frac{e^{it}}{(t-z)^\rho}$ is analytic on $\mathrm{Im}(t) > -\varepsilon$, for some $0 < \varepsilon < |\mathrm{Im}(z)|$, hence we have

$$0 = \int_{\Gamma_R} \frac{e^{it}}{(t-z)^\rho} dt = \int_{-R}^{R} \frac{e^{it}}{(t-z)^\rho} dt + \int_{\mathcal{C}_R} \frac{e^{it}}{(t-z)^\rho} dt.$$

Let $t = Re^{i\theta}$, then

$$\int_{\mathcal{C}_R} \frac{e^{it}}{(t-z)^\rho} dt = \int_0^\pi \frac{e^{-R\sin(\theta)}}{(Re^{i\theta} - z)^\rho} iRe^{i\theta} e^{iR\cos(\theta)} d\theta,$$

this converges to 0, as $R \to \infty$ by the dominated convergence theorem. It follows that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{it}}{(t-z)^\rho} dt = 0.$$

Case 2: $\mathrm{Im}(z) > 0$***.*** We integrate along the contour $\Gamma_{\varepsilon,\delta,R}$ presented in Figure 4.2.

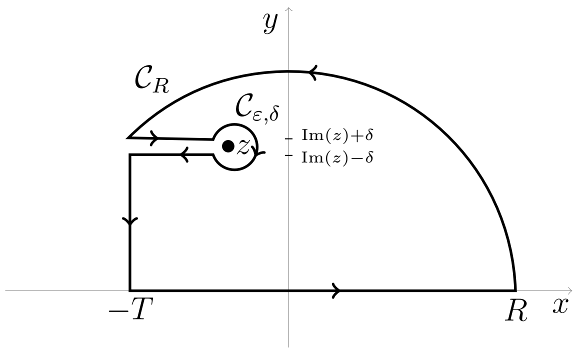

Figure 4.2: Contour $\Gamma_{\varepsilon,\delta,R}$

We set $z = x + iy$, and we have

$$\int_{\Gamma_{\varepsilon,\delta,R}} \frac{e^{it}}{(t-z)^\rho} dt = \int_{-T+(y+\delta)i}^{x+(y+\delta)i} \frac{e^{it}}{(t-z)^\rho} dt + \int_{\mathcal{C}_{\varepsilon,\delta}} \frac{e^{it}}{(t-z)^\rho} dt + \int_{x+(y-\delta)i}^{-T+(y-\delta)i} \frac{e^{it}}{(t-z)^\rho} dt$$
$$+ \int_{-T+iy}^{-T} \frac{e^{it}}{(t-z)^\rho} dt + \int_{-T}^{R} \frac{e^{it}}{(t-z)^\rho} dt + \int_{\mathcal{C}_R} \frac{e^{it}}{(t-z)^\rho} dt. \quad (4.1.1)$$

We analyse the integrals in turn. We have

$$\int_{-T+(y+\delta)i}^{x+(y+\delta)i} \frac{e^{it}}{(t-z)^\rho} dt \to \int_{-T}^{x} \frac{e^{i(u+iy)}}{|u+iy-z|^\rho e^{i\rho \arg(u+iy-z)}} du = \int_{-T}^{x} \frac{e^{i(u+iy)}}{(x-u)^\rho e^{i\rho\pi}} du; \quad (4.1.2)$$

$$\int_{x+(y-\delta)i}^{-T+(y-\delta)i} \frac{e^{it}}{(t-z)^\rho} dt \to \int_{x}^{-T} \frac{e^{i(u+iy)}}{|u+iy-z|^\rho e^{i\rho \arg(u+iy-z)}} du = \int_{-T}^{x} \frac{e^{i(u+iy)}}{(x-u)^\rho e^{-i\rho\pi}} du, \quad (4.1.3)$$

as $\delta \to 0$. Moreover,

$$\int_{\mathcal{C}_R} \frac{e^{it}}{(t-z)^\rho} dt \to 0, \text{ as } R \to \infty, \quad (4.1.4)$$

as in the Case 1. Also

$$\left| \int_{\mathcal{C}_{\varepsilon,\delta}} \frac{e^{it}}{(t-z)^\rho} dt \right| \le \left| \int_{-\pi}^{\pi} \frac{e^{i(z+\varepsilon e^{i\theta})}}{(\varepsilon e^{i\theta})^\rho} i\varepsilon e^{i\theta} d\theta \right| \ll \varepsilon^{1-\rho} \to 0, \quad (4.1.5)$$

as $\varepsilon \to 0$. Finally,

$$\int_{-T+iy}^{-T} \frac{e^{it}}{(t-z)^\rho} dt = \int_0^y \frac{e^{i(-T+iv)}}{(-T+iv-z)^\rho} i dv \to 0, \tag{4.1.6}$$

as $T \to \infty$. Since $\int_{\Gamma_{\varepsilon,\delta,R}} \frac{e^{it}}{(t-z)^\rho} dt = 0$ for all $R > 0$, $\delta > 0$ and $\varepsilon > 0$, by substituting (4.1.2)-(4.1.6) into (4.1.1), and letting $R \to \infty$, $\delta \to 0$ and $\varepsilon \to 0$, we deduce

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{it}}{(t-z)^\rho} dt &= \int_x^{-\infty} \frac{e^{i(u+iy)}}{(x-u)^\rho e^{i\rho\pi}} du - \int_x^{-\infty} \frac{e^{i(u+iy)}}{(x-u)^\rho e^{-i\rho\pi}} du \\ &= (e^{i\rho\pi} - e^{-i\rho\pi}) \int_0^\infty \frac{e^{i(x+iy)} e^{-iv}}{v^\rho} dv = 2i \sin(\pi\rho) e^{iz} \int_0^\infty v^{-\rho} e^{-iv} dv.\end{aligned}$$

It is a standard result, that if $0 < \rho < 1$, then

$$\int_0^\infty v^{-\rho} e^{-iv} dv = -ie^{i\rho\pi/2} \Gamma(1-\rho),$$

see for example [GR07, p. 346], Equation 7 with $k = 1 - \rho$, and $\mu = 1$. Summing up and using the identity $\Gamma(\rho)\Gamma(1-\rho) = \frac{\pi}{\sin(\rho\pi)}$, we have

$$\int_{-\infty}^{\infty} \frac{e^{it}}{(t-z)^\rho} dt = 2\sin(\pi\rho) e^{iz} e^{i\rho\pi/2} \Gamma(1-\rho) = \frac{2\pi e^{i\rho\pi/2}}{\Gamma(\rho)} e^{iz}.$$

This completes the proof of Claim 4.5. □

Claim 4.6. Let $\rho \in \mathbb{R}^+ \setminus \mathbb{Z}$, and z such that $\mathrm{Im}(z) \neq 0$. Then

$$\int_{-\infty}^{\infty} \frac{e^{it}}{(t-z)^\rho} dt = \begin{cases} \frac{2\pi e^{i\pi\rho/2}}{\Gamma(\rho)} e^{iz} & \text{if} \quad \mathrm{Im}(z) > 0, \\ 0 & \text{if} \quad \mathrm{Im}(z) < 0. \end{cases} \tag{4.1.7}$$

Proof. We will prove the claim by induction. Let $\alpha \in \mathbb{N}$, and $\rho \in (0,1)$, $R > 0$, integrating by parts gives:

$$\int_{-R}^{R} \frac{e^{it}}{(t-z)^\rho} = \frac{\rho}{i} \int_{-R}^{R} \frac{e^{it}}{(t-z)^{\rho+1}} + \left[\frac{e^{it}}{i(t-z)^\rho}\right]_{-R}^{R}.$$

Letting $R \to \infty$

$$\lim_{R\to\infty} \left[\frac{e^{it}}{i(t-z)^\rho}\right]_{-R}^{R} = 0,$$

and, by Claim 4.5,

$$\int_{-\infty}^{\infty} \frac{e^{it}}{(t-z)^{\rho+1}} dt = \begin{cases} \frac{2i\pi e^{i\pi\rho/2}}{\rho\Gamma(\rho)} e^{iz} = \frac{2\pi e^{i\pi(\rho+1)/2}}{\Gamma(\rho+1)} e^{iz} & \text{if} \quad \mathrm{Im}(z) > 0, \\ 0 & \text{if} \quad \mathrm{Im}(z) < 0. \end{cases}$$

This shows that formula (4.1.7) holds for $\rho \in (1,2)$. Let $\rho \in (k, k+1)$, and assume Equation (4.1.7) is true for all $n < k$, in particular for $\rho - 1$. Integrating by parts, we have

$$\int_{-R}^{R} \frac{e^{it}}{(t-z)^{\rho-1}} = \frac{\rho-1}{i} \int_{-R}^{R} \frac{e^{it}}{(t-z)^{\rho}} + \left[\frac{e^{it}}{i(t-z)^{\rho-1}}\right]_{-R}^{R}.$$

Again, letting $R \to \infty$ we have $\lim_{R\to\infty} \left[\frac{e^{it}}{i(t-z)^{\rho-1}}\right]_{-R}^{R} = 0$, and using the inductive hypothesis, we get

$$\int_{-\infty}^{\infty} \frac{e^{it}}{(t-z)^{\rho}} dt = \begin{cases} \frac{2\pi e^{i\pi\rho/2}}{\Gamma(\rho)} e^{iz} & \text{if} \quad \mathrm{Im}(z) > 0, \\ 0 & \text{if} \quad \mathrm{Im}(z) < 0. \end{cases}$$

This completes the proof of Claim 4.6. □

Claim 4.7. Let $\rho \in \mathbb{N}^+$, and z such that $\mathrm{Im}(z) \neq 0$. Then

$$\int_{-\infty}^{\infty} \frac{e^{it}}{(t-z)^{\rho}} dt = \begin{cases} \frac{2\pi e^{i\pi\rho/2}}{\Gamma(\rho)} e^{iz} & \text{if} \quad \mathrm{Im}(z) > 0, \\ 0 & \text{if} \quad \mathrm{Im}(z) < 0. \end{cases}$$

Proof. If $\mathrm{Im}(z) < 0$, then we repeat the proof from Claim 4.5. Assuming that $\mathrm{Im}(z) > 0$, we integrate along the contour Γ_R presented in Figure 4.3.

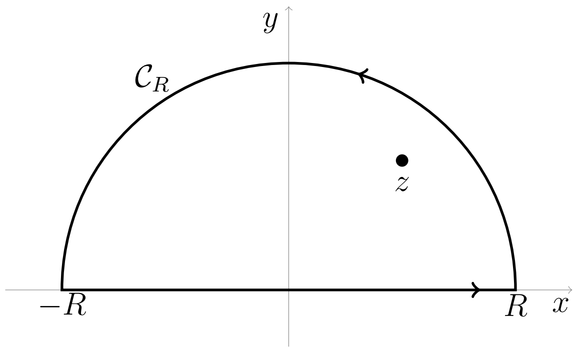

Figure 4.3: Contour Γ_R

We then have

$$\int_{-R}^{R} \frac{e^{it}}{(t-z)^{\rho}} dt + \int_{\mathcal{C}_R} \frac{e^{it}}{(t-z)^{\rho}} dt = 2i\pi \mathrm{Res}\left(\frac{e^{it}}{(t-z)^{\rho}}, t = z\right).$$

We have that

$$\lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{e^{it}}{(t-z)^{\rho}} dt = 0,$$

and also

$$\frac{e^{it}}{(t-z)^{\rho}} = e^{iz} \frac{e^{it-z}}{(t-z)^{\rho}} = e^{iz} \sum_{k=0}^{\infty} \frac{i^k}{k!} (t-z)^{k-\rho}.$$

Thus the residue for $k = \rho - 1$ is $e^{iz} \frac{i^{\rho-1}}{(\rho-1)!} = e^{iz} \frac{i^{\rho-1}}{\Gamma(\rho)}$. The result follows. □

4.2 Hölder regularity exponent of $M_{k,s}$ and $N_{k,s}$

4.2.1 Wavelet transform of $M_{k,s}$

Before we calculate the wavelet transform, we show the convergence of $M_{k,s}$ for certain s.

Lemma 4.8. *The series $M_{k,s}$ converges normally on $\mathbb{R}$ for all $s > k$, and for all $s > \frac{k}{2} + 1$ if M_k is a cusp form. Moreover, if M_k is a cusp form, then $M_{k,s}$ is well-defined and continuous on $\mathbb{R}$ for all $s > \frac{k}{2}$.*

Proof. By a result of Hecke, if M_k is not a cusp form, then $r_n = O(n^{k-1})$, and if M_k is a cusp form, then $r_n = O(n^{k/2})$, which proves the first part of the Lemma. Then Deligne proved that if M_k is a cusp form, then $r_n = O(n^{(k-1)/2+\varepsilon})$, for all $\varepsilon > 0$. For details, see for example [Ser73, p. 153-154]. Chamizo improved Deligne's result, showing that if M_k is a cusp form, then $M_{k,s}$ is well-defined and continuous on $\mathbb{R}$ for all $s > \frac{k}{2}$, [Cha04, Proposition 3.1]. □

We also need the following fact in order to calculate the wavelet transform of $M_{k,s}$.

Lemma 4.9. *Let $\rho > 1$, then*

$$\int_{\mathbb{R}} \frac{\sin(t)}{(t-z)^{\rho}} dt = \begin{cases} \frac{\pi e^{i\pi(\rho-1)/2}}{\Gamma(\rho)} e^{iz} & \text{if} \quad \mathrm{Im}(z) > 0, \\ \frac{\pi e^{-i\pi(\rho-1)/2}}{\Gamma(\rho)} e^{-iz} & \text{if} \quad \mathrm{Im}(z) < 0. \end{cases}$$

Proof. Recall that by Lemma 4.1, we have

$$\int_{\mathbb{R}} \frac{e^{it}}{(t-z)^{\rho}} dt = \begin{cases} \widetilde{c}(\rho) e^{iz} & \text{if} \quad \mathrm{Im}(z) > 0, \\ 0 & \text{if} \quad \mathrm{Im}(z) < 0. \end{cases}$$

On the other hand

$$\int_{\mathbb{R}} \frac{e^{it}}{(t-z)^{\rho}} dt = \int_{\mathbb{R}} \frac{e^{-iu}}{(-u-z)^{\rho}} du = \begin{cases} e^{i\pi\rho} \int_{\mathbb{R}} \frac{e^{-iu}}{(u+z)^{\rho}} du & \text{if} \quad \mathrm{Im}(z) > 0, \\ e^{-i\pi\rho} \int_{\mathbb{R}} \frac{e^{-iu}}{(u+z)^{\rho}} du & \text{if} \quad \mathrm{Im}(z) < 0. \end{cases}$$

Thus

$$\int_{\mathbb{R}} \frac{e^{-iu}}{(u-z)^{\rho}} du = \begin{cases} 0 & \text{if} \quad \mathrm{Im}(z) > 0, \\ \widetilde{c}(\rho) e^{-i\pi\rho} e^{-iz} & \text{if} \quad \mathrm{Im}(z) < 0. \end{cases}$$

The result then follows from $\sin(u) = \frac{e^{iu} - e^{-iu}}{2i}$. □

Now we will calculate the wavelet coefficients of $M_{k,s}$ with respect to the wavelet ψ_s.

Lemma 4.10. *Let $s > \frac{k}{2} + 1$ if M_k is a cusp form and $s > k$ otherwise. The wavelet transform of $M_{k,s}$ with respect to the wavelet ψ_s is*

$$C(a,b)(M_{k,s}) = \widehat{C} a^s (M_k(b+ia) - r_0),$$

where $\widehat{C} = (2\pi)^s \frac{\pi e^{i\pi s/2}}{\Gamma(s+1)}$. In particular, if M_k is a cusp form, then

$$C(a,b)(M_{k,s}) = \widehat{C} a^s M_k(b+ia).$$

Proof. By Lemma 4.8 we can change the the order of summation and integration, and we have

$$C(a,b)(M_{k,s}) = \frac{1}{a}\int_{\mathbb{R}} M_{k,s}(x)\frac{1}{\left(\frac{x-b}{a}-i\right)^{s+1}}dx = \frac{1}{a}\sum_{n=1}^{\infty}\frac{r_n}{n^s}\int_{\mathbb{R}}\frac{\sin(2\pi nx)}{\left(\frac{x-b}{a}-i\right)^{s+1}}dx.$$

Then we use the substitution $u = 2\pi nx$, and we obtain

$$\begin{aligned} C(a,b)(M_{k,s}) &= \frac{1}{a}\sum_{n=1}^{\infty}\frac{r_n}{n^s}\int_{\mathbb{R}}\frac{\sin(u)}{\left(\frac{\frac{u}{2\pi n}-b}{a}-i\right)^{s+1}}\frac{du}{2\pi n} \\ &= \frac{1}{a}\sum_{n=1}^{\infty}\frac{r_n}{n^s}\int_{\mathbb{R}}\frac{a^{s+1}(2\pi)^{s+1}n^{s+1}\sin(u)}{(u-2\pi nb-2\pi nai)^{s+1}}\frac{du}{2\pi n} \\ &= \frac{1}{a}\sum_{n=1}^{\infty}\frac{r_n}{n^s}\int_{\mathbb{R}}\frac{a^{s+1}(2\pi)^{s+1}n^{s+1}\sin(u)}{(u-2\pi nb-2\pi nai)^{s+1}}\frac{du}{2\pi n} \\ &= (2\pi)^s a^s\sum_{n=1}^{\infty} r_n\int_{\mathbb{R}}\frac{\sin(u)}{(u-2\pi nb-2\pi nai)^{s+1}}du. \end{aligned}$$

Then, by Lemma 4.9 we have

$$C(a,b)(M_{k,s}) = a^s(2\pi)^s\widetilde{c}(s+1)\sum_{n=1}^{\infty} r_n e^{2\pi in(b+ia)} = \widehat{C}a^s(M_k(b+ia)-r_0),$$

with $\widehat{C} = (2\pi)^s\widetilde{c}(s+1) = (2\pi)^s\frac{\pi e^{i\pi s/2}}{\Gamma(s+1)}$. □

The result is still valid if we let $M_k = E_2$. If $M_k = E_k$, then $F_{k,s}(x) = -\frac{B_k}{2k}M_{k,s}$. Therefore for all $k \in \mathbb{N}^*$ even, the wavelet transform of $F_{k,s}$ with respect to the wavelet ψ_s is

$$C(a,b)(F_{k,s}) = a^s c_k(s)(1-E_k(b+ia)),$$

with $c_k(s) = \frac{e^{i\pi s/2}2^{s-1}\pi^{s+1}B_k}{k\Gamma(s+1)}$.

We will now estimate the value of $C(a,b)(M_{k,s})$ distinguishing between cusp form and non-cusp form, and the case of E_2.

4.2.2 Estimating $C(a,b)(M_{k,s})$ when M_k is not a cusp form

We first estimate $|M_k(z)|$.

Claim 4.11. Let M_k be a modular form, not a cusp form. There exist $r, c_1, c_2, c_3 > 0$, such that:

if $\mathrm{Im}(z) \le r$, then

$$|M_k(z)| \le \frac{c_1}{\mathrm{Im}(z)^k};$$

if $\mathrm{Im}(z) \geq r$, then

$$c_2 \leq |M_k(z)| \leq c_3.$$

Proof. Let $r > 0$ such that $|r_0| > \sum_{n=1}^{\infty} |r_n| e^{-2\pi n r}$. Again, by Hecke we have $r_n = O(n^{k-1})$ (see [Ser73, p. 153-154]). Therefore, there exists $c_{1,k} > 0$ such that $|M_k(z)| \leq |r_0| + c_{1,k} \sum_{n=1}^{\infty} n^{k-1} e^{-2\pi n \mathrm{Im}(z)}$. Then there exists a polynomial P_{k-1} of degree $k-1$ vanishing at 0 such that $\sum_{n=1}^{\infty} n^{k-1} e^{-2\pi n \mathrm{Im}(z)} = \frac{P_{k-1}(e^{-2\pi \mathrm{Im}(z)})}{(1-e^{-2\pi \mathrm{Im}(z)})^k}$. Since $0 < e^{-2\pi \mathrm{Im}(z)} < 1$, there exists $c_{2,k} > 0$ such that $|P_{k-1}(e^{-2\pi \mathrm{Im}(z)})| \leq c_{2,k} e^{-2\pi \mathrm{Im}(z)}$. Finally, there exists $c_{3,k} > 0$ such that $\frac{e^{-2\pi \mathrm{Im}(z)}}{(1-e^{-2\pi \mathrm{Im}(z)})^k} \leq \frac{c_{3,k}}{(2\pi \mathrm{Im}(z))^k}$. Summing up, we get that

$$|M_k(z)| \leq |r_0| + \frac{c_{1,k} c_{2,k} c_{3,k}}{(2\pi)^k} \frac{1}{\mathrm{Im}(z)^k}.$$

If $\mathrm{Im}(z) \leq r$, then the first part of the Claim follows from setting $c_1 = |r_0| r^k + \frac{c_{1,k} c_{2,k} c_{3,k}}{(2\pi)^k}$.

On the other hand, if $\mathrm{Im}(z) \geq r$, then letting $c_2 = |r_0| - \sum_{n=1}^{\infty} |r_n| e^{-2\pi n r}$ and $c_3 = |r_0| + \sum_{n=1}^{\infty} |r_n| e^{-2\pi n r}$ gives the result. □

We define half-rings around the point x, see (4.2.1) below and Figure 4.4, and estimate the size of $|C(a,b)(M_{k,s})|$ using Claim 4.11. The following proposition is an analogue of Proposition 2 in [Jaf96]. The significant difference is that Jaffard fixes $D = 3$. This is possible because in the analogue of Claim 4.11 he can take $r = 1$. We cannot do it in general. In order to be able to use the lower bound from Claim 4.11, we need to carefully choose D, as we will see in the proof of the Proposition.

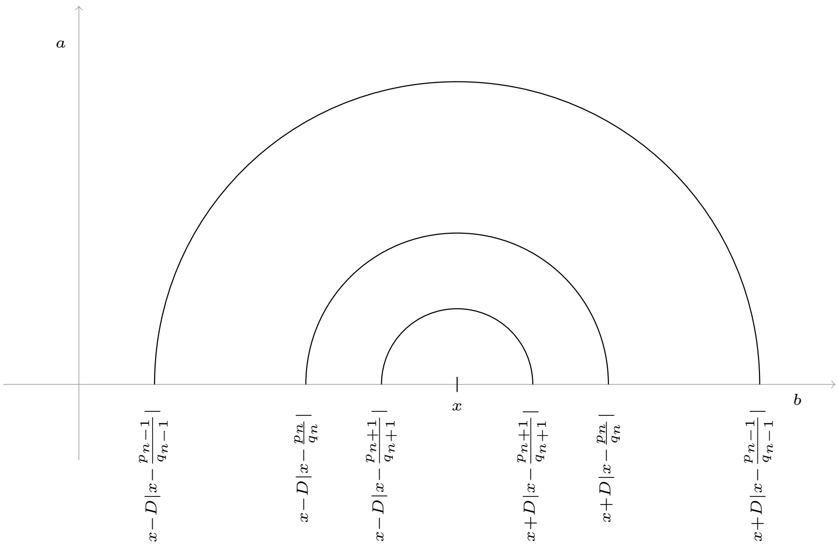

Figure 4.4: Half-rings around x

Proposition 4.12. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $a \in (0,1), b \in \mathbb{R}$.*

(i) Let $D > 1$. For each n, if

$$D\left|x - \frac{p_n}{q_n}\right| \leq |b - x + ia| \leq D\left|x - \frac{p_{n-1}}{q_{n-1}}\right|, \tag{4.2.1}$$

we have either:

$$|C(a,b)(M_{k,s})| \leq Ca^{s-k+k/\kappa_{n-1}}\left(1 + \frac{|b-x|}{a}\right)^{k/\kappa_{n-1}},$$

or

$$|C(a,b)(M_{k,s})| \leq Ca^{s-k+k/\kappa_n}\left(1 + \frac{|b-x|}{a}\right)^{k/\kappa_n},$$

for a constant C that may depend on k, s, x and D.

(ii) There exists $D_0 > 1$ depending at most on k, s and x, and there exists $\widetilde{C} > 0$ that may depend on k, s, x and D_0 such that for infinitely many n, there exists a point $b + ia$ in the domain (4.2.1) with $D = D_0$ satisfying

$$|C(a,b)(M_{k,s})| \geq \widetilde{C}a^{s-k+k/\kappa_n}\left(1 + \frac{|b-x|}{a}\right)^{k/\kappa_n}.$$

The first part of the proposition is later used to find the lower bound for the Hölder regularity exponent of $M_{k,s}$ at x, whereas we use the second part to find the upper bound for it, see the proof of Theorem 1.6.

Proof. Consider

$$\gamma_n = \begin{pmatrix} (-1)^n q_{n-1} & (-1)^{n-1} p_{n-1} \\ q_n & -p_n \end{pmatrix}.$$

By (2.1.1) we have $\gamma_n \in SL_2(\mathbb{Z})$. Let $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$. We have

$$\begin{aligned}
\gamma_n \cdot \left(\frac{p_n}{q_n} + z\right) &= \frac{(-1)^n q_{n-1}\left(\frac{p_n}{q_n} + z\right) + (-1)^{n-1} p_{n-1}}{q_n\left(\frac{p_n}{q_n} + z\right) - p_n} \\
&= \frac{(-1)^n (p_n q_{n-1} - p_{n-1} q_n) + z(-1)^n q_{n-1} q_n}{q_n^2 z} \\
&= \frac{-1 + (-1)^n z q_{n-1} q_n}{q_n^2 z} \\
&= \frac{(-1)^n q_{n-1}}{q_n} - \frac{1}{q_n^2 z}.
\end{aligned}$$

By (1.0.1) we have

$$\left|M_k\left(\frac{p_n}{q_n}+z\right)\right| = \frac{\left|M_k\left(\gamma_n\cdot\left(\frac{p_n}{q_n}+z\right)\right)\right|}{\left|q_n\left(\frac{p_n}{q_n}+z\right)-p_n\right|^k} = \frac{\left|M_k\left(\frac{(-1)^n q_{n-1}}{q_n}-\frac{1}{q_n^2 z}\right)\right|}{|q_n z|^k}. \tag{4.2.2}$$

Then we observe that

$$\operatorname{Im}\left(\frac{(-1)^n q_{n-1}}{q_n}-\frac{1}{q_n^2 z}\right) = \frac{\operatorname{Im}(z)}{q_n^2|z|^2}.$$

We now consider two cases.

Case 1: Assume that $\frac{\operatorname{Im}(z)}{q_n^2|z|^2}\le r$. By Claim 4.11 we have

$$\left|M_k\left(\frac{p_n}{q_n}+z\right)\right| \le c_1\frac{|q_n^2 z^2|^k}{|q_n z|^k \operatorname{Im}(z)^k} = c_1\frac{q_n^k|z|^k}{\operatorname{Im}(z)^k}.$$

For $z=b+ia-\frac{p_n}{q_n}$, we have

$$|M_k\left(b+ia\right)| \le c_1 a^{-k} q_n^k\left|b+ia-\frac{p_n}{q_n}\right|^k.$$

By (4.2.1), we have

$$\frac{D-1}{D}|b+ia-x| \le \left|b+ia-\frac{p_n}{q_n}\right| \le \frac{D+1}{D}|b+ia-x|. \tag{4.2.3}$$

Also by (4.2.1) and Lemma 2.4, since

$$\frac{D}{q_n^{\kappa_n}} \le |b+ia-x| \le \frac{D}{q_{n-1}^{\kappa_{n-1}}}$$

we have

$$|b+ia-x|^{-1/\kappa_n} \le q_n \le D|b+ia-x|^{(1-\kappa_{n-1})/\kappa_{n-1}}. \tag{4.2.4}$$

Substituting it, we get

$$\begin{aligned}|M_k\left(b+ia\right)| &\le c_1(D+1)^k a^{-k}|b+ia-x|^{k/\kappa_{n-1}}\\ &\le c_1(D+1)^k a^{-k+k/\kappa_{n-1}}\left(1+\frac{|b-x|}{a}\right)^{k/\kappa_{n-1}}.\end{aligned} \tag{4.2.5}$$

Since $-k+k/\kappa_{n-1}<0$ and $a\in(0,1)$, we have $a^{-k+k/\kappa_{n-1}}>1$. Also, as $k/\kappa_{n-1}>0$, we have $\left(1+\frac{|x-b|}{a}\right)^{k/\kappa_{n-1}}>1$. Then

$$|C(a,b)(M_{k,s})| = |a^s c_k(s)(1-M_k(b+ia))| \le a^s|c_k(s)|(1+|M_k(b+ia)|)$$

$$\leq a^s|c_k(s)|\left(1 + c_1(D+1)^k a^{-k+k/\kappa_{n-1}}\left(1+\frac{|b-x|}{a}\right)^{k/\kappa_{n-1}}\right)$$
$$\leq a^s|c_k(s)|\left(1 + c_1(D+1)^k\right) a^{-k+k/\kappa_{n-1}}\left(1+\frac{|b-x|}{a}\right)^{k/\kappa_{n-1}}.$$

The result follows with $C = |c_k(s)|\left(1 + c_1(D+1)^k\right)$.

Case 2: Assume that $\frac{\mathrm{Im}(z)}{q_n^2|z|^2} > r$. By Claim 4.11 we have

$$\left|M_k\left(\frac{p_n}{q_n}+z\right)\right| \leq \frac{c_3}{|q_n z|^k}.$$

By (4.2.4) and (4.2.3), we get

$$\begin{aligned} |M_k\left(b+ia\right)| &\leq c_3\left(\frac{D}{D-1}\right)^k |b+ia-x|^{-k+k/\kappa_n} \leq c_3\left(\frac{D}{D-1}\right)^k a^{-k+k/\kappa_n} \\ &\leq c_3\left(\frac{D}{D-1}\right)^k a^{-k+k/\kappa_n}\left(1+\frac{|x-b|}{a}\right)^{k/\kappa_n}. \end{aligned} \tag{4.2.6}$$

As before, since $a^{-k+k/\kappa_n}\left(1+\frac{|x-b|}{a}\right)^{k/\kappa_n} > 1$, we have

$$\begin{aligned} |C(a,b)(M_{k,s})| &= |a^s c_k(s)(1 - M_k(b+ia))| \leq a^s|c_k(s)|(1+|M_k(b+ia)|) \\ &\leq a^s|c_k(s)|\left(1 + c_3\left(\frac{D}{D-1}\right)^k a^{-k+k/\kappa_n}\left(1+\frac{|x-b|}{a}\right)^{k/\kappa_n}\right) \\ &\leq a^s|c_k(s)|\left(1 + c_3\left(\frac{D}{D-1}\right)^k\right) a^{-k+k/\kappa_n}\left(1+\frac{|x-b|}{a}\right)^{k/\kappa_n}. \end{aligned}$$

The result follows with $C = |c_k(s)|\left(1 + c_3\left(\frac{D}{D-1}\right)^k\right)$.

For the second part of Proposition 4.12, first suppose that $(q_n^{\kappa_n-2})_n$ is unbounded. Then for any $D > 1$ there exists an increasing sequence $(n_m)_m$, such that for all m we have

$$q_{n_m}^{\kappa_{n_m}-2} > \frac{(D^2+1)}{D} r, \tag{4.2.7}$$

where r is the constant defined in Claim 4.11, and n_m is large enough so that $q_{n_m}^{k\kappa_{n_m}-k} > 4(\sqrt{D^2+1})^k$. Now consider the point $a = \frac{D}{q_{n_m}^{\kappa_{n_m}}}$, $b = x$, which satisfies (4.2.1). Then we see that with $z = b + ia - \frac{p_{n_m}}{q_{n_m}}$,

$$|z| = \frac{\sqrt{D^2+1}}{q_{n_m}^{\kappa_{n_m}}},$$

and it follows that

$$\frac{\mathrm{Im}(z)}{q_n^2|z|^2} = \frac{D}{D^2+1} q_n^{\kappa_n - 2}.$$

Then by Claim 4.11, we have

$$\left| M_k \left(\frac{(-1)^{n_m} q_{n_m - 1}}{q_{n_m}} - \frac{1}{q_{n_m}^2 z} \right) \right| \geq c_2.$$

By (4.2.2) we have

$$\left| M_k \left(\frac{p_{n_m}}{q_{n_m}} + z \right) \right| \geq \frac{c_2}{|q_{n_m} z|^k} = \frac{c_2 q_{n_m}^{-k + k\kappa_{n_m}}}{(\sqrt{D^2+1})^k}, \tag{4.2.8}$$

and hence by (4.2.7), we have

$$\begin{aligned} |C(a,b)(M_{k,s})| &\geq c_k(s) \frac{c_2}{2(\sqrt{D^2+1})^k} q_n^{-s\kappa_n + k\kappa_n - k} \\ &\geq \widetilde{C} \left(\frac{D}{q_{n_m}^{\kappa_{n_m}}} \right)^{s - k + k/\kappa_{n_m}} \\ &= \widetilde{C}\,(a)^{s-k+k/\kappa_{n_m}} \left(1 + \frac{|b-x|}{a} \right)^{k/\kappa_{n_m}}, \end{aligned}$$

with $\widetilde{C} = \frac{c_2 c_k(s)}{2(\sqrt{D^2+1})^k D^{s-k+k/2}}$, and $D_0 = D$.

Now consider the second case, namely suppose that $(q_n^{\kappa_n - 2})_n$ is bounded. We will describe how we choose D_0. As $(q_n^{\kappa_n - 2})_n$ is bounded, it has a converging subsequence, and the limit L_0 is greater than or equal to 1, because $q_n \geq 1$ and $\kappa_n \geq 2$, for all n. Then

$$q_{n_\ell}^{\kappa_{n_\ell} - 2} \to L_0 \geq 1, \text{ as } \ell \to \infty. \tag{4.2.9}$$

We also observe that $\left(\frac{(-1)^{n_\ell} q_{n_\ell - 1}}{q_{n_\ell}} \right)_\ell$ is bounded, and has a converging subsequence. Suppose

$$\frac{(-1)^{n_{\ell(m)}} q_{n_{\ell(m)} - 1}}{q_{n_{\ell(m)}}} \to L_1. \tag{4.2.10}$$

Finally, since $(-1)^{n_{\ell(m)}} = 1$ for infinitely many m or $(-1)^{n_{\ell(m)}} = -1$ for infinitely many m, we may extract a constant subsequence of $(-1)^{n_{\ell(m)}}$. We will thus assume that all the elements are equal to 1, the same arguments apply to the other case. For simplicity we will denote this subsequence $(n_m)_m$.

Since M_k is a holomorphic function in $\mathbb{H}$, we can choose $D_0 > 1$ and $\delta > 0$ such that

$$M_k \left(L_1 - \frac{1 - iD_0}{D_0^2 + 1} L_0 \right) \neq 0, \tag{4.2.11}$$

and

$$\left|M_k\left(L_1 - \frac{1-iD_0}{D_0^2+1}L_0 + \varepsilon\right)\right| > \delta, \tag{4.2.12}$$

for ε small enough. Let $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$, (4.2.12) is satisfied. For each m consider the point

$$a = \frac{D_0}{q_{n_m}^{\kappa_{n_m}}}; b = x,$$

which satisfies (4.2.1). Using the previous notation $z = b + ia - \frac{p_{n_m}}{q_{n_m}}$, we have

$$\begin{aligned}
\frac{(-1)^{n_m} q_{n_m - 1}}{q_{n_m}} - \frac{1}{q_{n_m}^2 z} &= \frac{q_{n_m-1}}{q_{n_m}} - \frac{1}{q_{n_m}^2\left(\frac{D_0}{q_{n_m}^{\kappa_{n_m}}} i + x - \frac{p_{n_m}}{q_{n_m}}\right)} \\
&= \frac{q_{n_m-1}}{q_{n_m}} - \frac{1}{q_{n_m}^2\left(\frac{D_0}{q_{n_m}^{\kappa_{n_m}}} i + \frac{(-1)^{n_m}}{q_{n_m}^{\kappa_{n_m}}}\right)} = \frac{q_{n_m-1}}{q_{n_m}} - \frac{q_{n_m}^{\kappa_{n_m}-2}}{iD_0+1} \\
&= \frac{q_{n_m-1}}{q_{n_m}} - \frac{(1-iD_0) q_{n_m}^{\kappa_{n_m}-2}}{D_0^2+1} \to L_1 - \frac{1-iD_0}{D_0^2+1}L_0,
\end{aligned}$$

as $m \to \infty$, by (4.2.9) and (4.2.10). Therefore, there exists $L \in \mathbb{N}$ such that for all $m \ge L$ we have

$$\left|\frac{q_{n_m-1}}{q_{n_m}} - \frac{(1-iD_0) q_{n_m}^{\kappa_{n_m}-2}}{D_0^2+1} - L_1 + \frac{1-iD_0}{D_0^2+1}L_0\right| < \varepsilon_0,$$

and $q_{n_m}^{k\kappa_{n_m}-k} > \frac{2\left(\sqrt{D_0^2+1}\right)^k}{\delta}$. By (4.2.2) and (4.2.12) we have

$$\left|M_k\left(\frac{p_{n_m}}{q_{n_m}} + z\right)\right| \ge \frac{\delta}{|q_{n_m} z|^k} = \frac{\delta q_{n_m}^{-k+k\kappa_{n_m}}}{(\sqrt{D_0^2+1})^k}.$$

Then we have

$$\begin{aligned}
|C(a,b)(M_{k,s})| &\ge \frac{c_k(s)\delta}{2(\sqrt{D_0^2+1})^k} q_n^{-s\kappa_n + k\kappa_n - k} \ge \widetilde{C}\left(\frac{D_0}{q_{n_m}^{\kappa_{n_m}}}\right)^{s-k+k/\kappa_{n_m}} \\
&= \widetilde{C} a^{s-k+k/\kappa_{n_m}} \left(1 + \frac{|b-x|}{a}\right)^{k/\kappa_{n_m}},
\end{aligned}$$

with $\widetilde{C} = \frac{c_k(s)\delta}{2(\sqrt{D_0^2+1})^k D_0^{s-k+k/2}}$. This completes the proof of the proposition with D_0 satisfying (4.2.11). □

4.2.3 Estimating $C(a,b)(F_{2,s})$

We first estimate $|E_2(z)|$.

Claim 4.13. If $\mathrm{Im}(z) \leq 1$ then

$$|E_2(z)| \leq \frac{c_4}{\mathrm{Im}(z)^2},$$

for some $c_4 > 0$.

If $\mathrm{Im}(z) > 1$ then

$$\frac{19}{20} \leq |E_2(z)| \leq \frac{21}{20}.$$

Proof. We have

$$|E_2(z)| \leq 1 + 24\sum_{n=1}^{\infty} \sigma_1(n) e^{-2\pi n \mathrm{Im}(z)} \leq 1 + 24 \sum_{n \leq \frac{1}{\mathrm{Im}(z)}} \sigma_1(n) + 24 \sum_{j=1}^{\infty} e^{-2^j \pi} \sum_{n \leq \frac{2^j}{\mathrm{Im}(z)}} \sigma_1(n).$$

Since $\sum_{n=1}^{N} \sigma_1(n) = O(N^2)$, see for example [HW60, p. 266], it follows that there exists $c_4 > 0$ such that if $\mathrm{Im}(z) \leq 1$, then $|E_2(z)| \leq \frac{c_4}{\mathrm{Im}(z)^2}$.

The second part of the claim follows from

$$\left|24\sum_{n=1}^{\infty} \sigma_1(n) e^{2i\pi n z}\right| \leq 24 \sum_{n=1}^{\infty} \sigma_1(n) \left|e^{2i\pi n z}\right| \leq 24 \sum_{n=1}^{\infty} n^2 e^{-2\pi n \mathrm{Im}(z)} \leq 24 \sum_{n=1}^{\infty} n^2 e^{-2\pi n} \leq \frac{1}{20}.$$

□

Proposition 4.14. *Let $a \in (0,1), b \in \mathbb{R}$.*

(i) For each n, if

$$3\left|x - \frac{p_n}{q_n}\right| \leq |b - x + ia| \leq 3\left|x - \frac{p_{n-1}}{q_{n-1}}\right|, \tag{4.2.13}$$

we have either:

$$|C(a,b)(F_{2,s})| \leq C a^{s-2+2/\kappa_{n-1}} \left(1 + \frac{|b-x|}{a}\right)^{2/\kappa_{n-1}},$$

or

$$|C(a,b)(F_{2,s})| \leq C a^{s-2+2/\kappa_{n}} \left(1 + \frac{|b-x|}{a}\right)^{2/\kappa_{n}},$$

for a constant C that may depend on s and x.

(ii) Moreover, if for infinitely many n

$$a_n \geq 7, \tag{4.2.14}$$

then there exists $\widetilde{C} > 0$ that may depend on s and x, such that for infinitely many n (which satisfy (4.2.14)) there exists a point $b + ia$ in the domain (4.2.13) with

$$|C(a,b)(F_{2,s})| \geq \widetilde{C} a^{s-2+2/\kappa_{n-1}} \left(1 + \frac{|b-x|}{a}\right)^{2/\kappa_{n-1}}. \tag{4.2.15}$$

Proof. Consider

$$\gamma_n = \begin{pmatrix} (-1)^n q_{n-1} & (-1)^{n-1} p_{n-1} \\ q_n & -p_n \end{pmatrix} \in SL_2(\mathbb{Z}),$$

as in the proof of Proposition 4.12. By (2.2.1), we have

$$\begin{aligned} \left| E_2\left(\frac{p_n}{q_n} + z\right) \right| &\leq \frac{\left| E_2\left(\gamma_n \cdot \left(\frac{p_n}{q_n} + z\right)\right) \right|}{\left| q_n\left(\frac{p_n}{q_n} + z\right) - p_n \right|^2} + \frac{6}{\pi} \frac{q_n}{\left| q_n\left(\frac{p_n}{q_n} + z\right) - p_n \right|} \\ &= \frac{\left| E_2\left(\frac{(-1)^n q_{n-1}}{q_n} - \frac{1}{q_n^2 z}\right) \right|}{|q_n z|^2} + \frac{6}{\pi |z|}. \end{aligned} \tag{4.2.16}$$

Then we observe that

$$\operatorname{Im}\left(\frac{(-1)^n q_{n-1}}{q_n} - \frac{1}{q_n^2 z}\right) = \frac{\operatorname{Im}(z)}{q_n^2 |z|^2}.$$

Consider two cases.

Case 1: Assume that $\frac{\operatorname{Im}(z)}{q_n^2|z|^2} \leq 1$. We have

$$\frac{\left| E_2\left(\frac{(-1)^n q_{n-1}}{q_n} - \frac{1}{q_n^2 z}\right) \right|}{|q_n z|^2} \leq c_4 \frac{|q_n^2 z^2|^2}{|q_n z|^2 \operatorname{Im}(z)^2} = c_4 \frac{q_n^2 |z|^2}{\operatorname{Im}(z)^2}.$$

For $z = b + ia - \frac{p_n}{q_n}$, we have

$$|E_2(b+ia)| \leq c_4 a^{-2} q_n^2 \left| b + ia - \frac{p_n}{q_n} \right|^2 + \frac{6}{\pi \left| b + ia - \frac{p_n}{q_n} \right|}.$$

By (4.2.13), we have

$$\frac{2}{3}|b+ia-x| \leq \left| b + ia - \frac{p_n}{q_n} \right| \leq \frac{4}{3}|b+ia-x|.$$

Also by (4.2.13) and Lemma 2.4, since

$$\frac{3}{q_n^{\kappa_n}} \leq |b+ia-x| \leq \frac{3}{q_{n-1}^{\kappa_{n-1}}},$$

we have

$$|b+ia-x|^{-1/\kappa_n} \leq q_n \leq 3|b+ia-x|^{(1-\kappa_{n-1})/\kappa_{n-1}}.$$

Substituting it, we get

$$|E_2(b+ia)| \leq c_4 4^2 a^{-2} |b+ia-x|^{(2)/\kappa_{n-1}} + \frac{9}{\pi}|b+ia-x|^{-1}$$

$$\leq c_4 4^2 a^{-2+2/\kappa_{n-1}} \left(1 + \frac{|b-x|}{a}\right)^{2/\kappa_{n-1}} + \frac{9}{\pi} a^{-1}$$
$$\leq c_5 a^{-2+2/\kappa_{n-1}} \left(1 + \frac{|b-x|}{a}\right)^{2/\kappa_{n-1}},$$

for some constant c_5, because $-2 + 2/\kappa_{n-1} \leq -1$. Since $-2 + 2/\kappa_{n-1} < 1$ and $a \in (0,1)$, we have $a^{-2+2/\kappa_{n-1}} > 1$. Also, $2/\kappa_{n-1} > 0$ implies that $\left(1 + \frac{|b-x|}{a}\right)^{2/\kappa_{n-1}} > 1$. We have

$$\begin{aligned}
|C(a,b)(F_{2,s})| &= |a^s c_2(s)(1 - E_2(b+ia)| \leq a^s |c_2(s)|(1 + |E_2(b+ia)|) \\
&\leq a^s |c_2(s)| \left(1 + c_5 a^{-2+2/\kappa_{n-1}} \left(1 + \frac{|b-x|}{a}\right)^{2/\kappa_{n-1}}\right) \\
&\leq a^s |c_2(s)| \left(1 + c_5\right) a^{-2+2/\kappa_{n-1}} \left(1 + \frac{|b-x|}{a}\right)^{2/\kappa_{n-1}}.
\end{aligned}$$

The result follows with $C = |c_2(s)| \left(1 + c_5\right)$.

Case 2: Assume that $\frac{\mathrm{Im}(z)}{q_n^2|z|^2} > 1$. By Claim 4.13, we have

$$\left| E_2 \left(\frac{p_n}{q_n} + z \right) \right| \leq \frac{21}{20|q_n z|^2} + \frac{6}{\pi |z|}.$$

Using the same estimates as above, we get

$$\begin{aligned}
|E_2\left(b+ia\right)| &\leq \frac{21}{20}\left(\frac{3}{2}\right)^2 |b+ia-x|^{-2+2/\kappa_n} + \frac{9}{\pi|b+ia-x|} \\
&\leq \frac{189}{80} a^{-2+2/\kappa_n} + \frac{9}{\pi} a^{-1} \\
&\leq c_6 a^{-2+2/\kappa_n} \left(1 + \frac{|x-b|}{a}\right)^{2/\kappa_n},
\end{aligned}$$

fors some constant c_6, since $-2 + 2/\kappa_n \leq -1$. As $a^{-2+2/\kappa_n} \left(1 + \frac{|x-b|}{a}\right)^{2/\kappa_n} > 1$, we have

$$\begin{aligned}
|C(a,b)(F_{2,s})| &= |a^s c_2(s)(1 - E_2(b+ia)| \leq a^s |c_2(s)|(1 + |E_2(b+ia)|) \\
&\leq a^s |c_2(s)| \left(1 + c_6 a^{-2+2/\kappa_n} \left(1 + \frac{|x-b|}{a}\right)^{2/\kappa_n}\right) \\
&\leq a^s |c_2(s)| \left(1 + c_6\right) a^{-2+2/\kappa_n} \left(1 + \frac{|x-b|}{a}\right)^{2/\kappa_n}.
\end{aligned}$$

The result follows with $C = |c_2(s)| \left(1 + c_6\right)$.

We now prove (4.2.15). Consider the point $a = \frac{3}{q_n^{\kappa_n}}, b = x$, which satisfies (4.2.13). Then we note that with the notation $z = b + ia - \frac{p_n}{q_n}$, we have $|z| = \frac{\sqrt{10}}{q_n^{\kappa_n}}$, so that

$$\frac{\mathrm{Im}(z)}{q_n^2|z|^2} = \frac{3}{10} q_n^{\kappa_n - 2}.$$

We now assume that there exists an increasing sequence $(n_m)_m$, such that for all m we have $a_{n_m+1} \geq 7$. It follows that

$$q_{n_m}^{\kappa_{n_m}-2} = \frac{1}{q_{n_m}^2 \left| x - \frac{p_n}{q_n} \right|} \geq \frac{q_{n_m+1}}{q_{n_m}} > a_{n_m+1} \geq 7 > \frac{20\sqrt{10}}{3\pi}. \tag{4.2.17}$$

Then, $\frac{3}{10} q_{n_m}^{\kappa_{n_m}-2} \geq 1$, and by Claim 4.13, we have

$$\left| E_2 \left(\frac{(-1)^{n_m} q_{n_m-1}}{q_{n_m}} - \frac{1}{q_{n_m}^2 z} \right) \right| \geq \frac{19}{20}.$$

By (2.2.1) we have

$$\left| E_2 \left(\frac{p_{n_m}}{q_{n_m}} + z \right) \right| \geq \frac{19}{20|q_{n_m} z|^2} - \frac{6}{\pi|z|} = \frac{19 q_{n_m}^{-2+2\kappa_{n_m}}}{200} - \frac{6 q_{n_m}^{\kappa_{n_m}}}{\pi\sqrt{10}} \geq \frac{q_{n_m}^{-2+2\kappa_{n_m}}}{200}, \tag{4.2.18}$$

by our assumption (4.2.17). Furthermore, there exists $M \in \mathbb{N}$ such that for all $m \geq M$ we have $q_{n_m}^{2\kappa_{n_m}-2} > 400$, and we conclude that

$$\begin{aligned} |C(a,b)(F_{2,s})| &\geq c_2(s) \frac{1}{400} q_n^{-s\kappa_n + 2\kappa_n - 2} = c_2(s) \frac{1}{400} \frac{1}{3^{s-2+2/\kappa_{n_m}}} \left(\frac{3}{q_{n_m}^{\kappa_{n_m}}} \right)^{s-2+2/\kappa_{n_m}} \\ &\geq \widetilde{C} \left(\frac{3}{q_{n_m}^{\kappa_{n_m}}} \right)^{s-2+2/\kappa_{n_m}} = \widetilde{C} a^{s-2+2/\kappa_{n_m}} \left(1 + \frac{|b-x|}{a} \right)^{2/\kappa_{n_m}}, \end{aligned}$$

with $\widetilde{C} = c_2(s) \frac{1}{400} \frac{1}{3^{s-1}}$. □

As in the case of $k \geq 4$ we could show that even if the condition $a_{n_m+1} \geq 7$ is not satisfied, we could choose a sequence of points, each in the domain of (4.2.13), and $\delta > 0$ such that $\left| E_2 \left(\frac{(-1)^{n_m} q_{n_m-1}}{q_{n_m}} - \frac{1}{q_{n_m}^2 z} \right) \right| \geq \delta$. However, this is not enough to conclude that $|C(a,b)(F_{2,s})| \geq \widetilde{C} a^{s-2+2/\kappa_{n_m}} \left(1 + \frac{|b-x|}{a} \right)^{2/\kappa_{n_m}}$. Since $E_2(z)$ is quasimodular, there is an additional term $\frac{6}{\pi|z|}$ coming from (2.2.1), which has the same order of magnitude as $\frac{1}{|q_{n_m} z|^2}$ for our choice of z, and we do not see another choice of z that would counteract this extra term. Furthermore, in (4.2.18) we could use the other inequality, namely $|E_2(b+ia)| \geq \frac{6}{\pi|z|} - \frac{\left| E_2 \left(\frac{(-1)^n q_{n-1}}{q_n} - \frac{1}{q_n^2 z} \right) \right|}{|q_n z|^2}$, however the calculation then becomes considerably more technical and it would not solve all the remaining cases, namely the cases when $1 \leq a_n \leq 6$ for all $n \in \mathbb{N}$.

4.2.4 Estimating $C(a,b)(M_{k,s})$ when M_k is a cusp form

If M_k is a cusp form, $|M_k(z)|$ is not bounded below by a strictly positive constant as $\mathrm{Im}(z) \to \infty$. In the case of M_k not a cusp form, the lower bound was used to prove the optimality of the Hölder exponent. Therefore we add a condition on $(a_n)_n$ and $\mu(x)$.

Claim 4.15. Let M_k be a cusp form. There exists $c_1 > 0$, such that for all $z \in \mathbb{H}$ we have:

$$|M_k(z)| \leq \frac{c_1}{\mathrm{Im}(z)^{k/2+1}}.$$

Proof. By Hecke we have $r_n = O(n^{k/2})$ (see [Ser73, p. 153]). Therefore, there exists $c_{1,k} > 0$ such that $|M_k(z)| \leq c_{1,k} \sum_{n=1}^{\infty} n^{k/2} e^{-2\pi n \mathrm{Im}(z)}$. Then there exists a polynomial $P_{k/2}$ of degree $\frac{k}{2}$ vanishing at 0 such that $\sum_{n=1}^{\infty} n^{k/2} e^{-2\pi n \mathrm{Im}(z)} = \frac{P_{k/2}(e^{-2\pi \mathrm{Im}(z)})}{(1-e^{-2\pi \mathrm{Im}(z)})^{k/2+1}}$. Since $0 < e^{-2\pi \mathrm{Im}(z)} < 1$, there exists $c_{2,k} > 0$ such that $|P_{k/2}(e^{-2\pi \mathrm{Im}(z)})| \leq c_{2,k} e^{-2\pi \mathrm{Im}(z)}$. Finally, there exists $c_{3,k} > 0$ such that $\frac{e^{-2\pi \mathrm{Im}(z)}}{(1-e^{-2\pi \mathrm{Im}(z)})^{k/2+1}} \leq \frac{c_{3,k}}{(2\pi \mathrm{Im}(z))^{k/2+1}}$. ☐

We estimate $|C(a,b)(M_{k,s})|$ in the half-rings defined in (4.2.1) using Claim 4.15.

Proposition 4.16. *Let $k \geq 4$ be even. Let M_k be a cusp form of weight k, and let $s > \frac{k}{2}+1$. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $a \in (0,1), b \in \mathbb{R}$.*

(i) Let $D > 1$. For each n, if $b + ia$ satisfies (4.2.1), then we have:

$$|C(a,b)(M_{k,s})| \leq C a^{s-k/2-1+2/\kappa_{n-1}} \left(1 + \frac{|b-x|}{a}\right)^{2/\kappa_{n-1}},$$

for a constant C that may depend on k, s, x and D.

(ii) Moreover, let us assume that there exists $N \in \mathbb{N}$ such that for infinitely many n,

$$a_n = N. \tag{4.2.19}$$

Then, there exists $\widetilde{C} > 0$ that may depend on k, s and x, such that for an increasing subsequence of n, there exists a point $b + ia$ in the domain (4.2.1) with

$$|C(a,b)(M_{k,s})| \geq \widetilde{C} a^{s-k+k/\kappa_{n-1}} \left(1 + \frac{|b-x|}{a}\right)^{k/\kappa_{n-1}}.$$

Proof. As in the proof of Propostion 4.12, consider

$$\gamma_n = \begin{pmatrix} (-1)^n q_{n-1} & (-1)^{n-1} p_{n-1} \\ q_n & -p_n \end{pmatrix} \in SL_2(\mathbb{Z}).$$

As in (4.2.2), we have

$$\left| M_k \left(\frac{p_n}{q_n} + z \right) \right| = \frac{\left| M_k \left(\frac{(-1)^n q_{n-1}}{q_n} - \frac{1}{q_n^2 z} \right) \right|}{|q_n z|^k}. \tag{4.2.20}$$

We observe that

$$\mathrm{Im}\left(\frac{(-1)^n q_{n-1}}{q_n} - \frac{1}{q_n^2 z}\right) = \frac{\mathrm{Im}(z)}{q_n^2|z|^2}.$$

Then by Claim 4.15 we have

$$\left|M_k\left(\frac{p_n}{q_n} + z\right)\right| \leq \frac{c_1(q_n^2|z|^2)^{k/2+1}}{|q_n z|^k \mathrm{Im}(z)^{k/2+1}} = c_1 \frac{q_n^2|z|^2}{\mathrm{Im}(z)^{k/2+1}}.$$

For $z = b + ia - \frac{p_n}{q_n}$, we have

$$|M_k\left(b+ia\right)| \leq c_1 a^{-k/2-1} q_n^2 \left|b + ia - \frac{p_n}{q_n}\right|^2.$$

By (4.2.3) and (4.2.4) we have

$$\begin{aligned} |M_k\left(b+ia\right)| &\leq c_1(D+1)^2 a^{-k/2-1}|b+ia-x|^{2/\kappa_{n-1}} \\ &\leq c_1(D+1)^2 a^{-k/2-1+2/\kappa_{n-1}} \left(1 + \frac{|b-x|}{a}\right)^{2/\kappa_{n-1}}. \end{aligned}$$

Then

$$|C(a,b)(M_{k,s})| = a^s|c_k(s)|\,|M_k\left(b+ia\right)| \leq |c_k(s)| c_1(D+1)^2 a^{s-k/2-1}|b+ia-x|^{2/\kappa_{n-1}}.$$

The result follows with $C = |c_k(s)|c_1(D+1)^2$.

For the second part of Proposition 4.16, suppose that $a_n = N$ for infinitely many n for some $N \in \mathbb{N}$. Since $q_n^{\kappa_n - 2} \leq 4a_{n+1}$ we have that $(q_n^{\kappa_n-2})_n$ has a converging subsequence. The results then follows from exactly the same arguments as in the proof of Proposition 4.12. □

4.2.5 Proofs of Theorems 1.6, 1.9 and 1.10

We will start by considering the first assumption of Proposition J. Recall that $\mu(x) = \limsup_{n\to\infty} \kappa_n(x)$ and $\nu(x) = \liminf_{n\to\infty} \kappa_n(x)$.

Lemma 4.17. *For all $k \in \mathbb{N}^*$ even, $s > k$ and $x \in \mathbb{R}$, there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$ we have*

$$|\psi_s(x)| + \left|\psi_s^{(1)}(x)\right| + ... + \left|\psi_s^{(m+1)}(x)\right| \leq \frac{c}{(|x|+1)^{m+2}},$$

with $m = \left[s - k + \frac{k}{\mu(x)-\delta}\right]$, for some constant c.

Proof. For all $x \in \mathbb{R}$, from $(|x|-1)^2 \geq 0$ we get

$$\frac{2^{1/2}}{|x|+1} \geq \frac{1}{(|x|^2+1)^{1/2}} = \frac{1}{|x+i|}.$$

Then we note that for all $n \in \mathbb{N}^*$ we have

$$\psi_s^{(n)}(x) = \frac{(s+1)(s+2)...(s+n)}{(x+i)^{s+1+n}}.$$

If $\delta_0 \leq 1$, then $s+1+n \leq m+2$, we have

$$|\psi_s^{(n)}(x)| \leq \frac{c}{(|x+1|)^{m+2}},$$

for all $n \in \mathbb{N}^*$ for some constant c. It suffices to show now that $|\psi_s(x)| \leq \frac{1}{(|x|+1)^{m+2}}$.

Let $\delta_0 < \frac{k\mu(x)-k-\{s\}\mu(x)}{k-\{s\}}$. Since $k-\{s\} > 0$ and $k\mu(x)-k-\{s\}\mu(x) > 0$ for all $x \in \mathbb{R}, s > k$ and $k \geq 2$, we have $\delta_0 > 0$. It follows that $\{s\} + \frac{(1-k)(\mu(x)-\delta_0)+k}{\mu(x)-\delta_0} < 1$, therefore $\left[s-k+\frac{k}{\mu(x)-\delta}\right] \leq s-1$ for all $\delta \leq \delta_0$, which completes the proof of the Lemma. □

The proofs of Theorems 1.6, 1.9 and 1.10 are very similar and follow from Lemma 4.17, Propositions 4.12, 4.14, 4.16 and Proposition J. Therefore we will only present the details of the proof of Theorem 1.6. For the convenience of the reader, we recall it.

Theorem 1.6. *Let $k \geq 4$, even, and M_k be a modular form of weight k under $SL_2(\mathbb{Z})$ not a cusp form. For $x \in \mathbb{R} \setminus \mathbb{Q}$, let $\alpha_{M_{k,s}}(x)$ be the Hölder regularity exponent of $M_{k,s}$ at x. Assume that*

$$s > k + \frac{k}{\nu(x)} - \frac{k}{\mu(x)}. \tag{1.2.4}$$

Then,

$$\alpha_{M_{k,s}}(x) = s - k + \frac{k}{\mu(x)}.$$

The same is true if we replace $M_{k,s}$ with $N_{k,s}$.

Proof. Let $x \in \mathbb{R} \setminus \mathbb{Q}$, and assume that $s > k + \frac{k}{\nu(x)} - \frac{k}{\mu(x)}$. Let δ_0 as in Lemma 4.17. Assume that $\mu(x) < \infty$, very similar arguments apply to the other case, and therefore we omit the details. There exists $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$ we have

$$s > k + \frac{k}{\nu(x)-\delta} - \frac{k}{\mu(x)+\delta}. \tag{4.2.21}$$

Let $0 < \delta < \min(\delta_0, \delta_1)$ be given. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\nu(x) - \delta \leq \kappa_n \leq \mu(x) + \delta. \tag{4.2.22}$$

Let $D > 1$ and let $\omega = b + ia \in \mathbb{H}$ be such that

$$|\omega - x| \leq D\left|x - \frac{p_N}{q_N}\right|.$$

As (4.2.1) defines half-rings around x converging to x (see Figure 4.4), there exists $n_\omega > N$ such that
$$D\left|x-\frac{p_{n_\omega}}{q_{n_\omega}}\right| \leq |\omega - x| \leq D\left|x-\frac{p_{n_\omega-1}}{q_{n_\omega-1}}\right|.$$

By Proposition 4.12 we have
$$|C(a,b)(M_{k,s})| \leq Ca^{s-k+k/\kappa_{n_\omega}}\left(1+\frac{|b-x|}{a}\right)^{k/\kappa_{n_\omega}}$$
or
$$|C(a,b)(M_{k,s})| \leq Ca^{s-k+k/\kappa_{n_\omega-1}}\left(1+\frac{|b-x|}{a}\right)^{k/\kappa_{n_\omega-1}}.$$
It follows from (4.2.22) that
$$|C(a,b)(M_{k,s})| \leq Ca^{s-k+k/(\mu(x)+\delta)}\left(1+\frac{|b-x|}{a}\right)^{k/(\nu(x)-\delta)}.$$
Then we conclude by (4.2.21) and Proposition J that $M_{k,s} \in C^{s-k+k/(\mu(x)+\delta)}$ at x. Letting $\delta \to 0$ shows that $\alpha_{M_{k,s}}(x) \geq s-k+\frac{k}{\mu(x)}$.

For the optimality of this exponent, we see that Proposition 4.12 (ii) implies that for each $\delta > 0$ there exists a point $b+ia$, arbitrarily close to x such that
$$|C(a,b)(M_{k,s})| \geq \tilde{C}a^{s-k+k/(\mu(x)-\delta)}\left(1+\frac{|b-x|}{a}\right)^{k/(\nu(x)+\delta)}.$$
By Proposition J, we conclude that $M_{k,s} \notin C^{s-k+k/(\mu(x)-\delta)}$ at x. Letting $\delta \to 0$ shows that
$$\alpha_{M_{k,s}}(x) = s-k+\frac{k}{\mu(x)}.$$
This completes the proof of the theorem. □

4.2.6 Proof of Theorem 1.8

In this section, we prove Theorem 1.8. For the convenience of the reader, we recall it.

Theorem 1.8. *Let $k \geq 4$, even, M_k be a modular form of weight k under $SL_2(\mathbb{Z})$ not a cusp form and $s > \frac{3k}{2}$. Let $\alpha_{M_{k,s}}(x)$ be the Hölder regularity exponent of $M_{k,s}$ at x. Then*
$$\dim_{\mathrm{H}}\{x \in \mathbb{R} | \alpha_{M_{k,s}}(x) = \alpha\} = \begin{cases} \frac{2}{k}\alpha - \frac{2}{k}s + 2, & \text{if } \alpha \in \left[s-k, s-\frac{k}{2}\right], \\ 0 \text{ or } -\infty, & \text{otherwise.} \end{cases}$$

Proof. It is well known that the set $E_\mu = \{x \in \mathbb{R} | \mu(x) \geq \mu\}$ has the Hausdorff dimension $\dim_{\mathrm{H}} E_\mu = \frac{2}{\mu}$, see for example [Fal03, p. 157]. It has been shown in [Jaf96] that $\dim_{\mathrm{H}} E_\mu \setminus \cup_{\mu'>\mu} E_{\mu'} = \dim_{\mathrm{H}} E_\mu = \dfrac{2}{\mu}$. The result follows by noting that $\dim_{\mathrm{H}} \mathbb{Q} = 0$. □

4.2.7 Substituting cosine for sine

Theorems 1.6-1.10 remain still valid if we replace $F_{k,s}$ with $G_{k,s}$ and $M_{k,s}$ with $N_{k,s}$, since the wavelet transform of the series with cosine function with respect to ψ_s differs from the wavelet transform of the series with sine only by a multiplicative constant. For example, the wavelet transform of $G_{k,s}$ with respect to ψ_s is

$$C(a,b)(G_{k,s}) = \frac{1}{a}\int_{\mathbb{R}} G_{k,s}(x)\frac{1}{\left(\frac{x-b}{a}-i\right)^{s+1}}dx = \frac{2^s\pi^{s+1}e^{i\pi(s+1)/2}}{\Gamma(s+1)}a^s\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2i\pi n(b+ia)}.$$

We see that it differs from the wavelet transform of $F_{k,s}$ only by a multiplicative constant.

4.3 Hölder regularity exponent of $S_{d,s}$ and $T_{d,s}$

Before we start proving the theorems we note that the series that we consider are well-defined and continuous on $\mathbb{R}$. It follows from the next lemma.

Lemma 4.18. *The series $S_{d,s}$ and $T_{d,s}$ converges normally on $\mathbb{R}$ for $s > \frac{d}{2}$.*

Proof. Recall that the series

$$\sum_{n_1,\dots,n_d\geq 1}\frac{1}{(n_1^2+\dots+n_d^2)^s},$$

converges if and only if $\mathrm{Re}(s) > d/2$. □

We then note that we can write $S_{d,s}$ (and similarly $T_{d,s}$) as

$$S_{d,s} = \sum_{\ell=1}^{\infty}\frac{R_d(\ell)}{\ell^s}\sin(\pi\ell x),$$

where $R_d(\ell)$ is a number of ways to write ℓ as a sum of d squares of positive numbers where the order matters.

We remark that Stéphane Jaffard in [Jaf96] considered the case when $d = s = 1$. He proved the following theorem.

Theorem J. *Let $x \in \mathbb{R}\setminus\mathbb{Q}$, then*

$$\alpha_{S_{1,1}}(x) = \frac{1}{2} + \frac{1}{2\mu_e(x)}.$$

Also

$$\dim_{\mathrm{H}}\{x\in\mathbb{R}|\alpha_{S_{1,1}}(x)=\alpha\} = \begin{cases} 4\alpha-2, & \text{if } \alpha\in\left[\frac{1}{2},\frac{3}{4}\right], \\ 0, & \text{if } \alpha = \frac{3}{2}, \\ -\infty, & \text{otherwise.}\end{cases}$$

Our theorems are the generalisation of the first part of this result.

4.3.1 Wavelet transform of $S_{d,s}$

We start by computing the wavelet coefficients of $S_{d,s}$ with respect to ψ_s.

Lemma 4.19. *Let $s > \frac{d}{2}$. For $a > 0$ and $b \in \mathbb{R}$ we have*

$$\frac{1}{a}\int_{\mathbb{R}} S_{d,s}(x)\overline{\psi}_s\left(\frac{x-b}{a}\right)dx = a^s c_d(s)\left(\theta(b+ia)-1\right)^d,$$

where $c_d(s) = \frac{\pi^{s+1}e^{i\pi s/2}}{2^{d+1}\Gamma(s+1)}$.

Proof. Since $S_{d,s}$ converges normally on $\mathbb{R}$, we have

$$\frac{1}{a}\int_{\mathbb{R}} S_{d,s}(x)\frac{1}{\left(\frac{x-b}{a}-i\right)^{s+1}}dx = \frac{1}{a}\sum_{n_1=1}^{\infty}\cdots\sum_{n_d=1}^{\infty}\frac{1}{(n_1^2+\ldots+n_d^2)^s}\int_{\mathbb{R}}\frac{\sin(\pi(n_1^2+\ldots+n_d^2)x)}{\left(\frac{x-b}{a}-i\right)^{s+1}}dx.$$

Then we use the substitution $u = \pi(n_1^2+\ldots+n_d^2)x$, and we obtain

$$\begin{aligned}
&\frac{1}{a}\int_{\mathbb{R}} S_{d,s}(x)\frac{1}{\left(\frac{x-b}{a}-i\right)^{s+1}}dx \\
&\quad= \frac{1}{a}\sum_{n_1=1}^{\infty}\cdots\sum_{n_d=1}^{\infty}\frac{1}{(n_1^2+\ldots+n_d^2)^s}\int_{\mathbb{R}}\frac{\sin(u)}{\left(\frac{\frac{u}{\pi(n_1^2+\ldots+n_d^2)}-b}{a}-i\right)^{s+1}}\frac{du}{\pi(n_1^2+\ldots+n_d^2)} \\
&= \frac{1}{a}\sum_{n_1=1}^{\infty}\cdots\sum_{n_d=1}^{\infty}\frac{1}{(n_1^2+\ldots+n_d^2)^s}\int_{\mathbb{R}}\frac{a^{s+1}\pi^{s+1}(n_1^2+\ldots+n_d^2)^{(s+1)}\sin(u)}{(u-\pi(n_1^2+\ldots+n_d^2)(b+ia))^{s+1}}\frac{du}{\pi(n_1^2+\ldots+n_d^2)} \\
&= a^s\pi^s\sum_{n_1=1}^{\infty}\cdots\sum_{n_d=1}^{\infty}\int_{\mathbb{R}}\frac{\sin(u)}{(u-\pi(n_1^2+\ldots+n_d^2)(b+ai))^{s+1}}du.
\end{aligned}$$

Then by Lemma 4.9 we have have

$$\begin{aligned}
\frac{1}{a}\int_{\mathbb{R}} S_{d,s}(x)\frac{1}{\left(\frac{x-b}{a}-i\right)^{s+1}}dx &= a^s\pi^s\frac{c(s+1)}{2i}\sum_{n_1=1}^{\infty}\cdots\sum_{n_d=1}^{\infty}e^{i\pi(n_1^2+\ldots+n_d^2)(b+ia)} \\
&= a^s c_d(s)\left(\theta(b+ia)-1\right)^d,
\end{aligned}$$

with $c_d(s) = \frac{\pi^{s+1}e^{i\pi s/2}}{2^{d+1}\Gamma(s+1)}$. ∎

Therefore, the wavelet transform of $S_{d,s}$ with respect to the wavelet ψ_s is

$$C(a,b)(S_{d,s}) = a^s c_d(s)\left(\theta(b+ia)-1\right)^d.$$

4.3.2 Estimating $C(a,b)(S_{d,s})$

In this section we estimate $|\theta(b+ia)-1|$, as $a\to 0$ and $b\to x$. Jaffard in [Jaf96] divided the plane into half-rings as in (4.2.1) with $D=3$ (see Figure 4.4). Then he stated in Proposition 2, that $|\theta(b+ia)-1|\le Ca^{1/(2\kappa_n)-1/2}\left(1+\frac{|b-x|}{a}\right)^{1/(2\kappa_n)}$ or $|\theta(b+ia)-1|\le Ca^{1/(2\kappa_{n-1})-1/2}\left(1+\frac{|b-x|}{a}\right)^{1/(2\kappa_{n-1})}$ depending on the parity of p_n and q_n for some constant C. However, when p_n and q_n are both odd there seems to be a technical problem. Namely, on page 456 Jaffard is writing that if p_n, q_n are both odd, $3\left|x-\frac{p_n}{q_n}\right|\le|ia+b-x|\le 3\left|x-\frac{p_{n-1}}{q_{n-1}}\right|$, then

$$\left|b+ia-\frac{p_n}{q_n}\right|\ge 3\left|\rho-\frac{p_n}{q_n}\right|. \tag{4.3.1}$$

Consider the example of $x=\frac{1}{e}$, $\frac{p_5}{q_5}=\frac{7}{19}$, $a=10^{-6}$, $b=0.37001$. Then it verifies all the assumption, but Inequality (4.3.1) is not satisfied. We do not see how to fix this problem easily. Because of that, we use estimations of the size of $|\theta(b+ia)-1|$ in regions which were proposed by Oppenheim in his thesis [Opp97].

Oppenheim defined the regions around 0 and then he used the estimations of $\theta(x+b+ia)$ for $b+ia$ in one of the regions, and looked at the estimate when $a\to 0$ and $b\to 0$. In order to be consistent with the previous sections, we will redefine these regions by shifting each element by x. In this way we would consider the estimate of $\theta(b+ia)$ as $a\to 0$ and $b\to x$.

We define a disk Γ_n centred at $\left(\frac{p_n}{q_n},\frac{1}{2q_n^2}\right)$ of radius $\frac{1}{2q_n^2}$, that is:

$$\Gamma_n=\left\{b+ia\in\mathbb{H}\quad\Big|\quad\left|b+ia-\frac{p_n}{q_n}-\frac{i}{2q_n^2}\right|\le\frac{1}{2q_n^2}\right\},$$

see Figure 4.5.

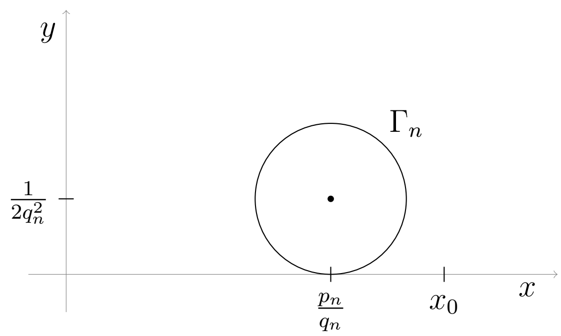

Figure 4.5: The contour of Γ_n

Also, for $n>m$ let

$$D(n,m)=\left\{b+ia\in\mathbb{H}\quad|\quad q_n\big(a^2+\big(b-\frac{p_n}{q_n}\big)^2\big)^{1/2}\le q_m\big(a^2+\big(b-\frac{p_m}{q_m}\big)^2\big)^{1/2}\right\}.$$

Then we define

$$\Omega_n = \Big(\bigcap_{0\le m<n} D(n,m)\Big) \setminus \Big(\bigcup_{m>n} D(m,n)\Big).$$

In other words,

$$\Omega_n = \bigcap_m \left\{ b+ia \in \mathbb{H} \quad \Big| \quad \frac{q_n\big(a^2+\big(b-\frac{p_n}{q_n}\big)^2\big)^{1/2}}{q_m\big(a^2+\big(b-\frac{p_m}{q_m}\big)^2\big)^{1/2}} \le 1 \right\}.$$

The contour of Ω_n depends of the values of a_n, a_{n+1}, a_{n+2} and all possible cases are presented in [Opp97, p. 84-85]. We have $\mathbb{H} = \cup_n(\Gamma_n \cup (\Omega_n \setminus \Gamma_n))$, $\Gamma_n \subset \Omega_n$ for all n, and as $n \to \infty$ the region Γ_n gets closer and closer to x. Then it is proved that

Proposition O. *[Opp97, p. 88, 92-94] Let $x \in \mathbb{R}\setminus\mathbb{Q}$. Let $\frac{1}{2} < t \le \frac{3}{4}$, $t+t' < \frac{1}{2}$. For all $n \in \mathbb{N}$ we have*

1. *if $b+ia \in \Gamma_n$ and p_n, q_n are not both odd, then*

$$|\theta(b+ia)| \le C'_n q_n^{-1/2}\big(a^2+\big(b-\frac{p_n}{q_n}\big)^2\big)^{-1/4} \le C_n a^{t-1}\left(1+\frac{|b-x|}{a}\right)^{-t'};$$

2. *if $b+ia \in \Gamma_n$ and p_n, q_n are both odd, then*

$$|\theta(b+ia)| \le C'_n q_n^{-1/2+2\sigma}\big(a^2+\big(b-\frac{p_n}{q_n}\big)^2\big)^{-1/4+\sigma} \le C_n a^{t-1}\left(1+\frac{|b-x|}{a}\right)^{-t'};$$

3. *if $b+ia \in \Omega_n \setminus \Gamma_n$, then*

$$|\theta(b+ia)| \le C'_n q_n^{-1/2}\big(a^2+\big(b-\frac{p_n}{q_n}\big)^2\big)^{-1/4} a^{-1/2} \le C_n a^{t-1}\left(1+\frac{|b-x|}{a}\right)^{-t'},$$

for some constants C'_n that depends on σ and n and C_n that depends on n.

On the other hand, if $b+ia \in \Gamma_n$ and p_n, q_n are not both odd, then

$$|\theta(b+ia)| \ge C' q_n^{-1/2}\big(a^2+\big(b-\frac{p_n}{q_n}\big)^2\big)^{-1/4},$$

for some C'.

Furthermore, $\limsup_{n\to\infty} C_n < \infty$ if and only if $t+t' \le \frac{1}{2}$ and

$$\limsup_{n\to\infty}\{q_n^{-1/2-\kappa_n/2+t\kappa_n} \quad | \quad p_n, q_n \text{ are not both odd}\} < \infty.$$

As $b \to x$ and $a \to 0$ we deduce that we can replace $\theta(b+ia)$ with $\theta(b+ia)-1$, only changing the constants.

4.3.3 Proof of Theorem 1.15

We will consider the first assumption of Proposition J.

Lemma 4.20. *For all* $d \in \mathbb{N}^*$, $s > \frac{d}{2}$ *and* $x \in \mathbb{R}$, *if* $\{s\} < \frac{d\mu_e(x)-d}{2\mu_e(x)}$, *then there exists* $\delta_0 > 0$ *such that for all* $0 < \delta \leq \delta_0$ *we have*

$$|\psi_s(x)| + \left|\psi_s^{(1)}(x)\right| + ... + \left|\psi_s^{(m+1)}(x)\right| \leq \frac{c}{(|x|+1)^{m+2}},$$

with $m = \left[s - \frac{d}{2} + \frac{d}{2\mu_e(x)-\delta}\right]$, *for some constant* c.

Proof. By Lemma 4.17 for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}^*$ we have

$$\psi_s^{(n)}(x) = \frac{(s+1)(s+2)...(s+n)}{(x+i)^{s+1+n}}.$$

If $\delta_0 \leq 1$, then $s+1+n \leq m+2$, we have

$$|\psi_s^{(n)}(x)| \leq \frac{c}{|x+1|)^{m+2}},$$

for all $n \in \mathbb{N}^*$ for some constant c. It suffices to show now that $|\psi_s(x)| \leq \frac{1}{(|x|+1)^{m+2}}$.

Let $\delta_0 < \frac{2d\mu_e(x)-2d-4\{s\}\mu_e(x)}{d-2\{s\}}$. Since $\{s\} < \frac{d\mu_e(x)-d}{2\mu_e(x)}$, we have $\delta_0 > 0$. It follows that $\{s\} + \frac{(2-d)(2\mu_e(x)-\delta_0)+2d}{2\mu_e(x)-\delta_0} < 1$, therefore $\left[s - \frac{d}{2} + \frac{d}{2\mu_e(x)-\delta}\right] \leq s-1$ for all $\delta \leq \delta_0$, which completes the proof of the Lemma. □

We are ready to prove Theorem 1.15. For the convenience of the reader, we recall it.

Theorem 1.15. *Let* $d \in \mathbb{N}^*$. *For* $x \in \mathbb{R} \setminus \mathbb{Q}$, *let* $\alpha_{S_{d,s}}(x)$ *be the Hölder regularity exponent of* $S_{d,s}$ *at* x. *Assume that*

$$s > \frac{d}{2} + \frac{d}{2\nu_e(x)} - \frac{d}{2\mu_e(x)}, \tag{1.2.9}$$

and

$$\{s\} < \frac{d\mu_e(x)-d}{2\mu_e(x)}, \tag{1.2.10}$$

then

$$\alpha_{S_{d,s}}(x) = s - \frac{d}{2} + \frac{d}{2\mu_e(x)}.$$

The same is true if we replace $S_{d,s}$ *with* $T_{d,s}$.

Proof. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Assume that $s > \frac{d}{2} + \frac{d}{2\nu_e(x)} - \frac{d}{2\mu_e(x)}$. Suppose that $\mu_e(x) < \infty$ the case when $\mu_e(x) = \infty$ is treated a very similar way and therefore omitted. There exists

$\delta_1 > 0$ such that for all $0 < \delta < \delta_1$, we have $s > \frac{d}{2} + \frac{d}{2\nu_{\mathrm{e}}(x)-\delta} - \frac{d}{2\mu_{\mathrm{e}}(x)+\delta}$. Let $0 < \delta < \delta_1$. Let $t = \frac{1}{2} + \frac{1}{2\mu_{\mathrm{e}}(x)+\delta}$ and $t' = -\frac{1}{2\nu_{\mathrm{e}}(x)-\delta}$. Then

$$t + t' = \frac{1}{2} + \frac{1}{2\mu_{\mathrm{e}}(x)+\delta} - \frac{1}{2\nu_{\mathrm{e}}(x)-\delta} \leq \frac{1}{2},$$

and

$$-\frac{1}{2} - \frac{\kappa_n}{2} + \big(\frac{1}{2} + \frac{1}{2\mu_{\mathrm{e}}(x)+\delta}\big)\kappa_n = -\frac{1}{2} + \frac{\kappa_n}{2\mu_{\mathrm{e}}(x)+\delta}.$$

Since there exists $N \in \mathbb{N}$ such that for all $n \geq N$ with p_n, q_n are not both odd, we have $\kappa_n < \mu_{\mathrm{e}}(x) + \frac{\delta}{2}$. Therefore $\limsup_{n\to\infty} \{q_n^{-1/2-\kappa_n/2+t\kappa_n} \mid p_n, q_n \text{ are not both odd}\} < \infty$. By Proposition O, we have

$$\begin{aligned}|C(a,b)(S_{d,s})| &\leq Ca^{s+(t-1)d}\left(1 + \frac{|b-x|}{a}\right)^{-t'd} \\ &= Ca^{s-d/2+d/(2\mu_{\mathrm{e}}(x)+\delta)}\left(1 + \frac{|b-x|}{a}\right)^{d/(2\nu_{\mathrm{e}}(x)-\delta)},\end{aligned}$$

for some constant C as $a \to 0$ and $b \to x$. Since $s - d/2 + d/(2\mu_{\mathrm{e}}(x) + \delta) < s - d/2 + d/(2\mu_{\mathrm{e}}(x) - \delta_0)$, by Lemma 4.20 we have that $\alpha = s - d/2 + d/(2\mu_{\mathrm{e}}(x) + \delta)$ fulfils the assumption of Proposition J. Since $s > \frac{d}{2} + \frac{d}{2\nu_{\mathrm{e}}(x)-\delta} - \frac{d}{2\mu_{\mathrm{e}}(x)+\delta}$ by Proposition J we have that $S_{d,s} \in C^{s-d/2+d/(2\mu_{\mathrm{e}}(x)+\delta)}$ at x. Letting $\delta \to 0$ shows that $\alpha_{S_{d,s}}(x) \geq s - \frac{d}{2} + \frac{d}{2\mu_{\mathrm{e}}(x)}$.

Taking $b + ia = \frac{p_n}{q_n} + \frac{i}{q_n^{\kappa_n}} \in \Gamma_n$ with p_n, q_n not both odd, by the second part of Proposition O we have that for $0 < \delta \leq \delta_0$ (where δ_0 is as defined in Lemma 4.20), there exists a point arbitrarily close to x such that

$$|C(a,b)(S_{d,s})| \geq \tilde{C}a^{s-d/2+d/(2\mu_{\mathrm{e}}(x)-\delta)}\left(1 + \frac{|b-x|}{a}\right)^{d/(2\nu_{\mathrm{e}}(x)+\delta)},$$

by Lemma 4.20 and Proposition J we conclude that $S_{d,s} \notin C^{s-d/2+d/(2\mu_{\mathrm{e}}(x)-\delta)}$ at x. Letting $\delta \to 0$ shows that $\alpha_{S_{d,s}}(x) = s - \frac{d}{2} + \frac{d}{2\mu_{\mathrm{e}}(x)}$. This completes the proof of the theorem. □

4.3.4 Proof of Theorem 1.17

In this section, we prove Theorem 1.17. For the convenience of the reader, we recall it.

Theorem 1.17. *Let $d \in \mathbb{N}, d \geq 4$. Let $s > \frac{3d}{4}$, then*

$$\dim_{\mathrm{H}}\{x \in \mathbb{R} | \alpha_{S_{d,s}}(x) = \alpha\} = \begin{cases} \frac{4}{d}\alpha - \frac{4}{d}s + 2, & \textit{if } \alpha \in \left[s - \frac{d}{2}, s - \frac{d}{4}\right], \\ 0 \textit{ or } -\infty, & \textit{otherwise.}\end{cases}$$

Proof. Let $E_{\mathrm{e},\mu} = \{x \in \mathbb{R} | \mu_{\mathrm{e}}(x) \geq \mu\}$. It has been shown in [Jaf96] that

$$\dim_{\mathrm{H}} E_{\mathrm{e},\mu} \setminus \bigcup_{\mu' > \mu} E_{\mathrm{e},\mu'} = \frac{2}{\mu}.$$

However, for the convenience of the reader, we present the main steps of the proof. Let $\mu_{\mathrm{o}}(x) = \limsup_{n\to\infty}\{\kappa_n(x) | p_n(x), q_n(x) \text{ are both odd}\}$, and let $E_{\mathrm{o},\mu} = \{x \in \mathbb{R} | \mu_{\mathrm{o}}(x) \geq \mu\}$; $E_\mu = \{x \in \mathbb{R} | \mu(x) \geq \mu\}$. We have

$$E_\mu = E_{\mathrm{e},\mu} \cup E_{\mathrm{o},\mu}.$$

By [Fal03, p.115,157], we have that $\dim_{\mathrm{H}} E_\mu = \frac{2}{\mu}$ and the $\frac{2}{\mu}$ dimensional Hausdorff meausure $\mathcal{H}^{2/\mu}$ of E_μ is positive. If the $\mathcal{H}^{2/\mu}$-measure of $E_{\mathrm{o},\mu}$ is zero, then the $\mathcal{H}^{2/\mu}$-measure of $E_{\mathrm{e},\mu}$ must be positive. On the other hand, if $x \in E_{\mathrm{o},\mu}$, then $\frac{x}{2} \in E_{\mathrm{e},\mu}$, thus if the $\mathcal{H}^{2/\mu}$-measure of $E_{\mathrm{o},\mu}$ is positive, so is the $\mathcal{H}^{2/\mu}$-measure of $E_{\mathrm{e},\mu}$. Finally, since the $\mathcal{H}^{2/\mu}$-measure of $\bigcup_{\mu'>\mu} E_{\mathrm{e},\mu'}$ is zero, $\dim_{\mathrm{H}} E_{\mathrm{e},\mu} \setminus \bigcup_{\mu'>\mu} E_{\mathrm{e},\mu'} = \frac{2}{\mu}$. The result follows by noting that $\dim_{\mathrm{H}} \mathbb{Q} = 0$. $\square$

Chapter 5

Conclusion

5.1 Alternative approaches

The differentiability of $G_{k,k+1}$ could be also studied using the connection to $((y))$ map. Let $((y)) = \{y\} - \frac{1}{2}$, its Fourier series is $((y)) = -\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\sin(2\pi my)}{m}$. Let $L_k(x) = 2\pi^2\sum_{r=1}^{\infty}\frac{((rx))}{r^k}$. It is an example of Davenport series which appear in the work of Hecke [Hec22]. The regularity of Davenport series was studied by Jaffard, see [Jaf04]. The function L_k converges uniformly on $\mathbb{R}$ and it is integrable. Then for $x \in \mathbb{R}$ we have

$$G_{k,k+1}(x) = \int_0^x L_k(t)dt + \zeta(2)\zeta(k+1).$$

We could then study the differentiability of $\int_0^x L_k(t)dt$. This approach was suggested to the author by Don Zagier.

Furthermore, the functions φ_k could be studied in the context of the theory of periods of modular forms. A period with moment s of a cusp form f of weight k introduced by Eichler is defined by $r_s(f) = \int_0^{i\infty} f(z)z^s dz$, for $0 \leq s \leq k-2$. This notion can be extended to modular forms, see [Zag91]. Stefano Marmi asked the question whether it is possible to obtain the functional equation of φ_k (3.1.48) using the methods presented in [Zag91] and [Lan76, Chapter V].

Another approach would be to analyse $M_{k,s}$ in a more general context, namely two-microlocal spaces $C^{s,s'}$, see for example [JM96, Jaf91], as it was done by Oppenheim for 2-dimensional Riemann function in [Opp97]. Let $s, s' \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, the two-microlocal space $C^{s,s'}_{x_0}$ is the Banach space of distributions such that $|S_0(f)(x)| \leq C(1 + |x - x_0|)^{-s'}$ and $|\Delta_j(f)(x)| \leq C2^{-js}(1 + 2^j|x - x_0|)^{-s'}$, where S_0 is a convolution operator. They generalise Hölder spaces in the sense that, for $s + s' > 0$ and $s > 0$ we have $C^{s,s'}(x_0) \subset C^s(x_0) \subset C^{s,-s}(x_0)$; and they can be characterised by wavelet transforms: a function f belongs to $C^{s,s'}_{x_0}$ if its wavelet transform satisfies $|C(a,b)(f)| \leq Ca^s\big(1 + \frac{|b-x_0|}{a}\big)^{-s'}$. They allow to investigate local behaviour of functions like chirps and logarithmic chirps, which describe strong local oscillations (for example of the type $x^\alpha \sin(x^{-\beta})$).

5.2 Further work

The method used in calculating Hölder regularity exponent of $M_{k,s}$ was only applicable for irrational points, as we used the infinite continued fraction expansion at x. In the study of rational points of the Riemann function, it was used that $\mathcal{S}$ is 2-periodic and satisfies: $\mathcal{S}(1+x) = \frac{1}{2}\mathcal{S}(4x) - \mathcal{S}(x)$. We do not have an analogue of such an equation in a general case. Therefore, one of the first questions to consider would be

Question 5.1. What is the Hölder regularity exponent of $M_{k,s}$ at a rational x?

Even though the functions $F_{k,s}$ and $G_{k,s}$ arose from Eisenstein series which exist only for k even, we could consider $F_{k,s}$ and $G_{k,s}$ for k odd. However, in this case we do not have the underlined modularity on which Itatsu's and Jaffard's methods were based in an essential way. New approaches would need to be developed.

The next step would be to generalise the findings to automorphic forms.

Question 5.2. Let A_k be an automorphic form of weight k under a Fuchsian group $G \subseteq SL_2(\mathbb{Z})$ of finite index such that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in G$ and having multiplier system m such that $m(\gamma) = 1$. It admits a Fourier expansion $A_k(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$, see for example [Iwa97, Section 2.7]. Define a series $A_{k,s}(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{2\pi i n x}$ for suitable s. Chamizo showed that if $s < \frac{k}{2} + 1$ then $A_{k,s}$ is not differentiable at any irrational number, [Cha04, Corollary 2.1.1]. He also showed that the Hölder regularity exponent of $A_{k,s}$ at irrational points is equal to $s - \frac{k}{2}$ for all $\frac{k}{2} < s < \frac{k}{2} + 1$ if A_k is a cusp form, [Cha04, Theorem 2.1]. What is the Hölder regularity exponent of $A_{k,s}$ at an irrational x for $s > \frac{k}{2} + 1$? What is the Hölder regularity exponent of $A_{k,s}$ at an irrational x if A_k is not a cusp form?

A further direction would be to consider functions of multiple variables depending on a parameter. For example, for $t \in [0,1]$ and and $x \in [0,2]$ we could study

$$F_s(x,t) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)\cos(2\pi n t)}{n^s}.$$

We note that $F_2(x,0)$ is the Riemann function. The procedure would be to use the method of Itatsu and find the functional equation exploiting the connection to $\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z + 2\pi i n \tau}$, which satisfies certain Jacobi identities. It is conjectured that $F_2(x, \frac{2}{3})$ is differentiable at $x = \frac{3}{4}$, for instance.

Another question that remains open is the behaviour of $S_{3,2}$ and $T_{3,2}$ at irrational points with $\mu_{\mathrm{e}}(x) = 3$. By Theorem 1.15, the Hölder regularity exponent at these points is equal to 1. Moreover, both functions are differentiable at irrationals such that $\mu_{\mathrm{e}}(x) < 3$ and not differentiable at irrationals such that $\mu_{\mathrm{e}}(x) > 3$, therefore we ask

Question 5.3. Are the functions $S_{3,2}$ and $T_{3,2}$ differentiable at irrational points with $\mu_{\mathrm{e}}(x) = 3$?

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Propriétés analytiques et diophantiennes de certaines séries de Fourier arithmétiques

Résumé: Nous considérons certaines séries de Fourier liées à la théorie des formes modulaires. Nous étudions leurs propriétés analytiques : la dérivabilité, le module de continuité et l'exposant de Hölder. Nous utilisons deux méthodes différentes. La première revient à trouver et itérer une équation fonctionnelle de la fonction étudiée (méthode d'Itatsu) et la deuxième provient de l'analyse en ondelettes (méthode de Jaffard). L'étape essentielle de chacune dépend de la modularité sous-jacente. Nous trouvons que les propriétés analytiques de ces séries aux points irrationnels sont liées aux propriétés diophantiennes de ces points. Ce travail a été motivé par l'étude de la fonction de Riemann.
Mots clés: dérivabilité, module de continuité, exposant de Hölder, formes modulaires, séries d'Eisenstein, fonction thêta, ondelettes, fractions continues

Analytic and Diophantine properties of certain arithmetic Fourier series

Abstract: We consider certain Fourier series which arise from modular or automorphic forms. We study their analytic properties: differentiability, modulus of continuity and the Hölder regularity exponent. We use two different methods. One is based on finding and iterating a functional equation for the function studied (Itatsu's method), the second one comes from wavelet analysis (Jaffard's method). The crucial steps in both of them are based on the underlined modularity. We find that the analytic properties of these series at an irrational x are related to the fine diophantine properties of x, in a very precise way. The work was motivated by the study of the Riemann series.
Keywords: Differentiability, Modulus of continuity, Hölder regularity, Modular forms, Eisenstein series, Theta function, Wavelets, Continued fractions